MAT 1341B: Introduction to Linear Algebra

Instructor: Erhard Neher

Final Exam (December 21, 2004)

Family Name: ______________________________

First Name: ______________________________

Student number: ___________________________

Please read these instructions carefully:

- This is a closed book exam, and no notes of any kind are allowed. **The use of calculators, cell phones, pagers or any text storage or communication device is not permitted, and will be considered as fraud.**
- Questions 1–10 are multiple choice questions, each worth 2 points. No part marks will be given. **Please record your answers on the line "My answer: ___________" provided for each multiple choice question.**
- Questions 11–15 are long-answer questions, worth 6–10 points. Partial marks may be earned. Please be careful to include all details, and explain what you are doing.
- The last question is a bonus "proof" question. You may earn 4 bonus points.
- Read each question carefully - you will save yourself time and unnecessary grief later on. Where it is possible to check your work, do so.
- If you do not have enough space, use the back of the pages and clearly indicate this. The exam has 15 pages. You have 180 minutes to complete this exam.

Good luck!

Quest.	total MC	11.	12.	13.	14.	15.	16.	Total
maximal	20	10	6	6	8	10	4 (bonus)	60
score								

1. (2 points) For which value(s) of a is the following linear system solvable?

$$\begin{array}{ccccc} x & + & 2y & = & 5 \\ 2x & + & 2y & = & 7 \\ x & + & ay & = & -1 \end{array}$$

A. for no value of a, i.e., the system is not solvable

B. $a = 3$ and $a = 4$

C. $a = 3$

D. $a = -2$

E. $a \neq 1$

F. $a = 0$ and $a = 5$

My answer:____________

2. (2 points) Let A be a 7×5 matrix, and suppose that the homogeneous linear system $AX = 0$ is uniquely solvable. Answer the following questions:

(i) What is the rank of A?

(ii) If the linear system $AX = B$ is solvable, is it then uniquely solvable?

A. $\operatorname{rank}(A) = 7$; no.

B. $\operatorname{rank}(A) = 5$; yes.

C. $\operatorname{rank}(A) = 7$; yes.

D. $\operatorname{rank}(A) = 2$; yes.

E. $\operatorname{rank}(A) = 5$; no.

F. $\operatorname{rank}(A) = 2$; no.

My answer:____________

3. (2 points) If

$$\left(A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

then $A =$

A. $\begin{bmatrix} 5 & 5 \\ 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 7 & 8 \\ 0 & 10 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 5 \\ 7 & 9 \end{bmatrix}$

D. $\begin{bmatrix} 4 & -4 \\ 5 & 11 \end{bmatrix}$

E. $\begin{bmatrix} 2 & -3 \\ -6 & 8 \end{bmatrix}$

F. $\begin{bmatrix} 8 & 2 \\ 7 & -5 \end{bmatrix}$

My answer:____________

4. (2 points) The determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

is

A. 165

B. 0

C. -35

D. 23

E. -75

F. 68

My answer:____________

5. (2 points) If z, w are solutions of the linear system

$$\begin{array}{rcrcl} z & + & (1+i)w & = & 2 \\ & & (1-i)w & = & 1+i \end{array}$$

then

A. $z = 1 + i$
B. $w = -i$
C. $z = 2 - i$
D. $w = 1 + 2i$
E. $z = 3 - i$
F. $z = 4 + 2i$

My answer:____________

6. (2 points) If we denote by C_1, C_2, C_3, C_4, C_5 the columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 2 & 6 & -5 & -2 & 4 \\ 0 & 0 & 5 & 10 & 0 \\ 2 & 6 & 0 & 8 & 4 \end{bmatrix}$$

then a basis of the column space of A is

A. C_1, C_2, C_3
B. C_1, C_3, C_4
C. C_1, C_3
D. C_1, C_2
E. C_1, C_3, C_5
F. C_1, C_4

My answer:____________

7. (2 points) Let U be a subspace of $\mathbb{R}^n$. Which of the following statements are true?

(i) If $X \in U^{\perp}$ then $\text{proj}_U(X) = 0$.

(ii) $\dim U = \dim U^{\perp}$.

(iii) If W is another subspace such that $W \subset U$ then $U^{\perp} \subset W^{\perp}$.

A. (i) and (ii)
B. (ii) only
C. (ii) and (iii)
D. (i) only
E. (i) and (iii)
F. (iii) only

My answer:____________

8. (2 points) Which of the following are subspaces?

$$U = \{A \in \mathbb{M}_{22} \mid \det(A) = 0\},$$
$$V = \{\, f \in \mathbb{F}[0,2] \mid f(x) = 0 \text{ for all } x \in [0,1]\},$$
$$W = \{\, p \in \mathbb{P}_3 \mid p(-1) = 1\}.$$

Reminder: $\mathbb{M}_{22}$ is the vector space of 2×2-matrices; $\mathbb{F}[0,2]$ is the vector space of functions $f\colon [0,2] \to \mathbb{R}$; $\mathbb{P}_3$ is the vector space consisting of the zero polynomial and all polynomials of degree ≤ 3.

A. V only.
B. U only.
C. W only.
D. V and W only.
E. U and W only.
F. U and V only.
G. All three of them.

My answer:____________

9. (2 points) Which of the following statements are true?

(i) If V is a vector space of dimension n, then every set of n linearly independent vectors in V is a spanning set of V.

(ii) $\mathbb{P}_2$ contains a basis of polynomials p satisfying $p(0) = 2$.

(iii) $\{1, \sin^2(x), \cos^2(x)\}$ are a linearly independent subset of $\mathbb{F}[0, 2\pi]$.

A. all of them
B. (i) and (iii)
C. (ii) and (iii)
D. Only (i)
E. Only (iii)
F. (i) and (ii)
G. Only (ii)
H. None of them

My answer:____________

10. (2 points) Let V be a vector space. Which of the following statements are true?

(i) If $\{u, v, w\}$ is a linearly independent subset of V, then also $\{u, v\}$ is linearly independent.

ii) Every spanning set of V contains a basis of V.

(iii) If $\dim(V) = n$ then every set of n linearly independent vectors of V is a basis.

A. all of them
B. (i) and (iii)
C. (ii) and (iii)
D. Only (i)
E. Only (iii)
F. (i) and (ii)
G. Only (ii)
H. None of them

My answer:____________

11. (10 points) Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -3 & -2 & -3 \\ -1 & 0 & 0 \end{bmatrix}.$$

(a) Find the characteristic polynomial $c_A(x) = \det(xI_3 - A)$, and conclude that the eigenvalues of A are -2 and 1.

(b) For each eigenvalue of A find a basis of the corresponding eigenspace.

(c) Decide if A is diagonalizable or not. Justify your answer. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(You have two pages to complete this problem.)

(continue on next page)

12. (6 points) Write down three different statements that are equivalent for an $n \times n$ matrix A to be invertible.

(I)

(II)

(III)

13. (6 points) The vectors

$$X_1 = [\,1\,,\,1\,,\,1\,,\,1\,], \quad X_2 = [\,1\,,\,0\,,\,0\,,\,1\,], \quad X_3 = [\,0\,,\,2\,,\,1\,,\,-1\,]$$

are a basis of a subspace U of $\mathbb{R}^4$. Find an orthogonal basis of U.

14. (8 points) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by

$$T[x_1\,,\, x_2\,,\, x_3]^T = [2x_1 - 3x_2 + 4x_3\,,\, -x_1 + x_2]^T$$

Show that T is a linear map and find its standard matrix. (You have two pages to complete this problem.)

(continue on next page)

15. (10 points) Recall that $\mathbb{P}_2$ is the vector space consisting of the zero polynomial and all polynomials of degree ≤ 2. Find a basis of the subspace $U = \{p \in \mathbb{P}_2 | p(2) = 0\}$ of $\mathbb{P}_2$, and determine $\dim U$. (You have two pages to complete this problem.)

(continue on next page)

16. (4 bonus points) Let V be a vector space, and let $\mathbf{u}, \mathbf{v} \in V$. Use the definition of a subspace to prove that $\mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{v} = \{a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ is a subspace.