MAT 1341B: Introduction to Linear Algebra

Instructor: Erhard Neher

Final Exam (December 21, 2004)

Family Name: ____________________________

First Name: ____________________________

Student number: ____________________________

Please read these instructions carefully:

- This is a closed book exam, and no notes of any kind are allowed. **The use of calculators, cell phones, pagers or any text storage or communication device is not permitted, and will be considered as fraud.**
- Questions 1–10 are multiple choice questions, each worth 2 points. No part marks will be given. **Please record your answers on the line "My answer: ___________" provided for each multiple choice question.**
- Questions 11–15 are long-answer questions, worth 6–10 points. Partial marks may be earned. Please be careful to include all details, and explain what you are doing.
- The last question is a bonus "proof" question. You may earn 4 bonus points.
- Read each question carefully - you will save yourself time and unnecessary grief later on. Where it is possible to check your work, do so.
- If you do not have enough space, use the back of the pages and clearly indicate this. The exam has 15 pages. You have 180 minutes to complete this exam.

Good luck!

Quest.	total MC	11.	12.	13.	14.	15.	16.	Total
maximal	20	10	6	6	8	10	4 (bonus)	60
score								

1. (2 points) For which value(s) of a is the following linear system solvable?

$$\begin{array}{rcrcr} x & + & 2y & = & 5 \\ 2x & + & 2y & = & 7 \\ x & + & ay & = & -1 \end{array}$$

A. for no value of a, i.e., the system is not solvable
B. $a = 3$ and $a = 4$
C. $a = 3$
D. $a = -2$
E. $a \neq 1$
F. $a = 0$ and $a = 5$

My answer:____________

Answer: We find the row-echelon form of the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 2 & 7 \\ 1 & a & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & -2 & -3 \\ 0 & a-2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & a & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & -3-\frac{3}{2}a \end{bmatrix}.$$

The system is therefore solvable if and only if $-3 - \frac{3}{2}a = 0 \iff 1 + \frac{a}{2} = 0 \iff a = -2$. The correct answer is therefore D.

2. (2 points) Let A be a 7×5 matrix, and suppose that the homogeneous linear system $AX = 0$ is uniquely solvable. Answer the following questions:

(i) What is the rank of A?
(ii) If the linear system $AX = B$ is solvable, is it then uniquely solvable?

A. rank $(A) = 7$; no.
B. rank $(A) = 5$; yes.
C. rank $(A) = 7$; yes.
D. rank $(A) = 2$; yes.
E. rank $(A) = 5$; no.
F. rank $(A) = 2$; no.

My answer:____________

Answer: By §1.3, Th. 2, the number of free parameters in the general solution is $n - r$, where n is the number of variables, that is $n = 5$ in our case, and r is the rank of A. Because $AX = 0$ is uniquely solvable, there are no free parameters, hence $n - r = 0$ or $r = n = 5$. Theorem 2 in §1.4 says that the system $AX = B$ is uniquely solvable if it is solvable at all. Thus, the correct answer is B.

3. (2 points) If

$$\left(A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

then $A =$

A. $\begin{bmatrix} 5 & 5 \\ 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 7 & 8 \\ 0 & 10 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 5 \\ 7 & 9 \end{bmatrix}$

D. $\begin{bmatrix} 4 & -4 \\ 5 & 11 \end{bmatrix}$

E. $\begin{bmatrix} 2 & -3 \\ -6 & 8 \end{bmatrix}$

F. $\begin{bmatrix} 8 & 2 \\ 7 & -5 \end{bmatrix}$

My answer:____________

Answer: We take the inverse on both sides and get

$$A^T - 3\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{hence}$$

$$A^T = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -4 & 11 \end{bmatrix} \quad \text{and then}$$

$$A = \begin{bmatrix} 4 & -4 \\ 5 & 11 \end{bmatrix}.$$

Thus the correct answer is D.

Reference: §1.5, suggested exercise 5; Problem session of Oct.8.

4. (2 points) The determinant of the matrix

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

is

A. 165

B. 0

C. -35

D. 23

E. -75

F. 68

My answer:____________

Answer: We calculate, using elementary row operations

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3)(-55) = 165.$$

In the last step we used that the determinant of a triangular matrix is the product of the diagonal elements. Thus, the correct answer is A.

Reference: §2.1, suggested exercises 3, 12, 13.

5. (2 points) If z, w are solutions of the linear system

$$\begin{array}{rcrcl} z & + & (1+i)w & = & 2 \\ & & (1-i)w & = & 1+i \end{array}$$

then

A. $z = 1+i$

B. $w = -i$

C. $z = 2-i$

D. $w = 1+2i$

E. $z = 3-i$

F. $z = 4+2i$

My answer:____________

Answer: From the second equation we get

$$w = \frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+2i+i^2}{1+1} = \frac{2i}{2} = i,$$

and then from the first equation $z = 2-(1+i)i = 1-i-i^2 = 2-i+1 = 3-i$. Hence the correct answer is E.

Reference: §2.5, suggested exercise 10; problem session of Oct. 29.

6. (2 points) If we denote by C_1, C_2, C_3, C_4, C_5 the columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 2 & 6 & -5 & -2 & 4 \\ 0 & 0 & 5 & 10 & 0 \\ 2 & 6 & 0 & 8 & 4 \end{bmatrix}$$

then a basis of the column space of A is

A. C_1, C_2, C_3

B. C_1, C_3, C_4

C. C_1, C_3

D. C_1, C_2

E. C_1, C_3, C_5

F. C_1, C_4

My answer:____________

Answer: The find a row-echelon form of A:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 2 & 6 & -5 & -2 & 4 \\ 0 & 0 & 5 & 10 & 0 \\ 2 & 6 & 0 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 5 & 10 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

By the Rank Theorem (§4.5, Th. 2), a basis of the column space is given by the columns corresponding to the columns with the leading $1's$ in R, that is the columns C_1 and C_3. Hence C is correct.

Here is another solution (suggested by a student): Since the $\mathrm{Col}\,(A) = \mathrm{Row}\,(A^T)$, we find the row-echelon form of A^T (switch row 2 and row 4, subtract multiplies of row 1 from row 3,4 and 5, subtract multiples of row 2 from row 3):

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 6 & 0 & 6 \\ -2 & -5 & 5 & 0 \\ 0 & -2 & 10 & 8 \\ 2 & 4 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 10 & 8 \\ 0 & -1 & 5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 10 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R'$$

Since R' is a row-echelon matrix, Th. 1 in §4.5 says that the first two rows of R', which are $[1 \quad 2 \quad 0 \quad 2]$ and $[0 \quad 1 \quad -5 \quad -4]$ are a basis of $\mathrm{Row}\,(A^T) = \mathrm{Col}\,(A)$. But one can multiply any basis vector by a non-zero scalar, hence also $[1 \quad 2 \quad 0 \quad 2]$ and $[0 \quad -2 \quad 10 \quad 8]$ are a basis of $\mathrm{Row}\,(A^T) = \mathrm{Col}\,(A)$. But these are columns C_1 and C_4 of the matrix A. So also F is a correct answer.

One see this also in a different manner, as follows. Form the row echelon form of A we get that $\dim \mathrm{Col}\,(A) = 2$. Hence every two linearly independent columns are a basis of $\mathrm{Col}\,(A)$. In particular, this rules out the answers A, B and E. Since $C_1 = 3C_2$, they are linearly dependent, hence D is not true either. It is easy to see that C_1, C_3 and C_1, C_4 are linearly independent. Therefore, both are bases of $\mathrm{Col}\,(A)$ (remember that a subspace has many different bases), and both C and F are correct answers and will be accepted as correct answers.

Reference: §4.4, suggested exercise 5; problem session of Nov. 12.

7. (2 points) Let U be a subspace of $\mathbb{R}^n$. Which of the following statements are true?

(i) If $X \in U^\perp$ then $\mathrm{proj}_U(X) = 0$.

(ii) $\dim U = \dim U^\perp$.

(iii) If W is another subspace such that $W \subset U$ then $U^\perp \subset W^\perp$.

A. (i) and (ii)
B. (ii) only
C. (ii) and (iii)
D. (i) only
E. (i) and (iii)
F. (iii) only

My answer:____________

Answer: (i) is true: Any vector can be uniquely decomposed as a sum of a vector in U and another vector in $U^\perp$. Since $X \in U^\perp$ this decomposition is $X = 0 + X$ in our case, so $\mathrm{proj}_U(X) = 0$. Another proof: $\mathrm{proj}_U(X) = 0$ in view of the formula for $\mathrm{proj}_U(X)$, see §4.6, Lemma 1. (ii) is false: the correct formula is $\dim U^\perp = n - \dim U$ (§4.6, Th. 3). (iii) is true: if $X \in U^\perp$ then $X \cdot Y = 0$ for all $Y \in U$, hence in particular for all $Y \in W$. But this just says that $X \in W^\perp$. Thus the correct answer is E.

Reference: §4.6, suggested exercise 9.

8. (2 points) Which of the following are subspaces?

$$U = \{A \in \mathbb{M}_{22} \mid \det(A) = 0\},$$
$$V = \{\, f \in \mathbb{F}[0,2] \mid f(x) = 0 \text{ for all } x \in [0,1]\},$$
$$W = \{\, p \in \mathbb{P}_3 \mid p(-1) = 1\}.$$

Reminder: $\mathbb{M}_{22}$ is the vector space of 2×2-matrices; $\mathbb{F}[0,2]$ is the vector space of functions $f\colon [0,2] \to \mathbb{R}$; $\mathbb{P}_3$ is the vector space consisting of the zero polynomial and all polynomials of degree ≤ 3.

A. V only.
B. U only.
C. W only.
D. V and W only.
E. U and W only.
F. U and V only.
G. All three of them.

My answer:____________

Answer: U is not a subspace; for example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in U \quad \text{but} \quad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin U.$$

The set V is a subspace: since $0_{\mathbb{F}[0,2]}$ is the function for which $f(x) = 0$ for *all* $x \in [0,2]$ we have $0_{\mathbb{F}[0,2]} \in V$. Also, by definition of addition in $\mathbb{F}[0,2]$, if $f, g \in V$ then $f+g$ has the property that $(f+g)(x) = f(x)+g(x) = 0 + 0 = 0$ for all $x \in [0,1]$. Finally, if $f \in V$ and $c \in \mathbb{R}$ then the function cf satisfies the condition of V: $(cf)(x) = cf(x) = c0 = 0$ for all $x \in [0,1]$. Thus, $cf \in V$, and V fulfills all conditions defining a subspace. The set W is not a subspace since, for example, the zero polynomial does not lie in W. The correct answer is therefore A.

9. (2 points) Which of the following statements are true?

(i) If V is a vector space of dimension n, then every set of n linearly independent vectors in V is a spanning set of V.

(ii) $\mathbb{P}_2$ contains a basis of polynomials p satisfying $p(0) = 2$.

(iii) $\{1, \sin^2(x), \cos^2(x)\}$ are a linearly independent subset of $\mathbb{F}[0, 2\pi]$.

A. all of them
B. (i) and (iii)
C. (ii) and (iii)
D. Only (i)
E. Only (iii)
F. (i) and (ii)
G. Only (ii)
H. None of them

My answer:____________

Answer: (i) is true by §5.2, Th 4. (ii) is true, for example the polynomials $2, x+2, x^2+2$ are such a basis. (iii) is false: $1 - \sin^2(x) - \cos^2(x)$ is a nontrivial linear combination. The correct answer is F.

Reference: suggested exercise §5.2, 1 and 2; problem session of Dec. 3.

10. (2 points) Let V be a vector space. Which of the following statements are true?

(i) If $\{u, v, w\}$ is a linearly independent subset of V, then also $\{u, v\}$ is linearly independent.

ii) Every spanning set of V contains a basis of V.

(iii) If $\dim(V) = n$ then every set of n linearly independent vectors of V is a basis.

A. all of them

B. (i) and (iii)

C. (ii) and (iii)

D. Only (i)

E. Only (iii)

F. (i) and (ii)

G. Only (ii)

H. None of them

My answer:____________

Answer: (i) is true: If $au + bv = 0_V$ for $a, b \in \mathbb{R}$ then also $au + bv + cw = 0_W$, hence $a = 0_{\mathbb{R}} = b$ since $\{u, v\}$ is linearly independent. (ii) is true: §5.2, Th. 3. (iii) is true: §5.2, Th 4. Thus, the correct answer is A.

11. (10 points) Let A be the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -3 & -2 & -3 \\ -1 & 0 & 0 \end{bmatrix}.$$

(a) Find the characteristic polynomial $c_A(x) = \det(xI_3 - A)$, and conclude that the eigenvalues of A are -2 and 1.

(b) For each eigenvalue of A find a basis of the corresponding eigenspace.

(c) Decide if A is diagonalizable or not. Justify your answer. If yes, give an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(You have two pages to complete this problem.)

Answer: (a) To find all eigenvalues we calculate the characteristic polynomial

$$\begin{aligned} c_A(x) = \det(xI_3 - A) = \det \begin{bmatrix} x-2 & 0 & -1 \\ 3 & x+2 & +3 \\ 1 & 0 & x \end{bmatrix} &= (x+2)\det \begin{bmatrix} x-2 & 0 & -1 \\ 3 & 1 & 3 \\ 1 & 0 & x \end{bmatrix} \\ = (x+2)\det \begin{bmatrix} x-2 & -1 \\ 1 & x \end{bmatrix} &= (x+2)(x^2 - 2x + 1) = (x+2)(x-1)^2 \\ = x^3 - 3x + 2. \end{aligned}$$

The eigenvalues are the roots of $c_A(x)$. Therefore the eigenvalues of A are -2 and 1.

(b) To find a basis of the eigenspace corresponding to the eigenvalue -2 we need to find a basis of the null space *null* $(-2I_3 - A)$, where I_3 is the 3×3-identity matrix. We do this by row reducing $-2I_3 - A$:

$$-2I_3 - A = \begin{bmatrix} -4 & 0 & -1 \\ 3 & 0 & 3 \\ 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 9 \\ 0 & 0 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the corresponding homogeneous linear system is $x_1 = 0 = x_3$ and its general solution is $x_1 = 0 = x_3$, $x_2 = t$, t a free parameter. Hence a basis of the eigenspace $E_{-2} = null\,(-2I_3 - A)$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Similarly, to find a basis of the eigenspace corresponding to the eigenvalue 1 we need to find a basis of the null space *null* $(I_3 - A)$, which we do by row reducing $-2I_3 - A$:

$$I_2 - A = \begin{bmatrix} -1 & 0 & -1 \\ 3 & 3 & 3 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the corresponding homogeneous linear system is $x_1 + x_3 = 0 = x_2$ and its general solution is $x_2 = 0$, $x_1 = t = -x_3$, t a free parameter. Hence a basis of the eigenspace $E_1 = null\,(I_3 - A)$ is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

(c) The matrix is not diagonalizable since the multiplicity of the eigenvalue 1 is 2, but the dimension of the eigenspace E_1 is only 1 (see §2.3, Th. 5).

Reference: §2.3, suggested exercise 2,4,6; problem session of Oct. 15

Marking: 4 points for (a): 1 point for knowing that the eigenvalues of A are the roots of the characteristic polynomial, 2 points for calculating the $c_A(x)$ correctly, 1 point for finding the roots. (b) 2 points for each of the two eigenspaces. (c) 2 points, depending on the answer in (b).

12. (6 points) Write down three different statements that are equivalent for an $n \times n$ matrix A to be invertible.

(I)
(II)
(III)

Answer: Any three of the 17 conditions of the Invertible Matrix Theorem (posted on the course web site) are correct:

(1) There is an $n \times n$-matrix C such that $CA = I_n$.
(2) There is an $n \times n$-matrix D such that $AD = I_n$.
(3) The reduced row echelon form of A is I_n.
(4) A has rank n.
(5) The linear system $AX = b$ has a unique solution for every column b.
(6) The linear system $AX = b$ has at least one solution for every column b.
(7) The homogeneous linear system $AX = 0$ has only the trivial solution, i.e., $\operatorname{null} A = \{0\}$ where $\operatorname{null} A$ is the null space of A.
(8) A^T is invertible.
(9) $\det(A) \neq 0$.
(10) 0 is not an eigenvalue of A.
(11) The columns of A are linearly independent.
(12) The columns of A are a basis of $\mathbb{R}^n$.
(13) The columns of A span $\mathbb{R}^n$, i.e., $\operatorname{col} A = \mathbb{R}^n$.
(14) The rows of A are linearly independent.
(15) The rows of A are a basis of $\mathbb{R}^n$.
(16) The rows of A span $\mathbb{R}^n$, i.e., $\operatorname{col} A = \mathbb{R}^n$.
(17) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$, given by $X \mapsto AX$, is invertible.

A typical wrong solution is "A does not have a row of zeros". This condition is necessary but not sufficient and hence not **equivalent** to A being invertible. For example, the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

does not have a row or column of zeros, yet it is not invertible.

Reference: This is the Invertible Matrix Theorem posted on the course web site.

Marking: 2 points per correct answer

13. (6 points) The vectors

$$X_1 = [\,1\,,\,1\,,\,1\,,\,1\,], \quad X_2 = [\,1\,,\,0\,,\,0\,,\,1\,], \quad X_3 = [\,0\,,\,2\,,\,1\,,\,-1\,]$$

are a basis of a subspace U of $\mathbb{R}^4$. Find an orthogonal basis of U.

Answer: We apply the Gram-Schmidt algorithm (§4.5, Th. 8) and find an orthogonal basis F_1, F_2, F_3 of U:

$$\begin{aligned}
F_1 &= X_1, \\
F_2 &= X_2 - \frac{X_2 \cdot F_1}{F_1 \cdot F_1} F_1 = [1\,,\,0\,,\,0\,,\,1] - \frac{2}{4}[1\,,\,1\,,\,1\,,\,1] = [\frac{1}{2}\,,\,-\frac{1}{2}\,,\,-\frac{1}{2}\,,\,\frac{1}{2}], \\
F_3 &= X_3 - \frac{X_3 \cdot F_1}{F_1 \cdot F_1} F_1 - \frac{X_3 \cdot F_2}{F_2 \cdot F_2} F_2 \\
&= [0\,,\,2\,,\,1\,,\,-1] - \frac{1}{2}[1\,,\,1\,,\,1\,,\,1] + 2[\frac{1}{2}\,,\,-\frac{1}{2}\,,\,-\frac{1}{2}\,,\,\frac{1}{2}] \\
&= [\frac{1}{2}\,,\,\frac{1}{2}\,,\,-\frac{1}{2}\,,\,-\frac{1}{2}].
\end{aligned}$$

Note that in an orthogonal basis you an multiply any vector with a non-zero scalar. The result will still be an orthogonal basis. Hence, for example

$$[1, 1, 1, 1],\ [1, -1, -1, 1],\ [1, 1, -1, -1]$$

is also an orthogonal basis of U. It is also not necessary to start with X_1. You can also start with X_2 or X_3. The Gram-Schmidt algorithm will then produce another orthogonal basis.

Reference: suggested exercise §4.5, 7bd; also 7c was done in the problem session of Nov.12.

Marking: 3 points for the correct formula for the vectors F_i, 3 points for the correct calculations.

14. (8 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by

$$T[x_1\,,\,x_2\,,\,x_3]^T = [2x_1 - 3x_2 + 4x_3\,,\,-x_1 + x_2]^T$$

Show that T is a linear map and find its standard matrix. (You have two pages to complete this problem.)

Answer: **Solution 1:** We check that the 2 conditions for a linear map. For $x = [x_1\,,\,x_2\,,\,x_3]^T$ and $y = [y_1\,,\,y_2\,,\,y_3]^T$ we have

$$\begin{aligned} T(x) + T(y) &= [2x_1 - 3x_2 + 4x_3\,,\,-x_1 + x_2]^T + [2y_1 - 3y_2 + 4y_3\,,\,-y_1 + y_2]^T \\ &= [2(x_1 + y_1) - 3(x_2 + y_2) + 4(x_3 + y_3)\,,\,-(x_1 + y_1) + (x_2 + y_2)]^T \\ &= T(x + y), \end{aligned}$$

and for $c \in \mathbb{R}$ we get $T(cx) = T[cx_1\,,\,cx_2\,,\,cx_3]^T = [2cx_1 - 3cx_2 + 4cx_3\,,\,-cx_1 + cx_2]^T = c[2x_1 - 3x_2 + 4x_3\,,\,-x_1 + x_2]^T = cT(x)$. This proves that T is linear. To find the standard matrix A of T we apply Th. 2 of §4.9. The standard matrix is the 2×3 matrix whose columns are $T(E_1), T(E_2)$ and $T(E_3)$ where E_1, E_2, E_3 is the standard basis of $\mathbb{R}^3$. We get

$$A = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution 2: One first shows that $T(x) = Ax$ (matrix multiplication), where A is the matrix above. This implies that T is linear (see Example 5 in §4.9), and then one checks that A is in fact the standard matrix of T (as mentioned in class).

Reference: suggested exercise §4.9, 4; a similar problem was done in the problem session of Nov.26.

Marking: 4 points for proving that T is linear, and 4 points for the standard matrix.

15. (10 points) Recall that $\mathbb{P}_2$ is the vector space consisting of the zero polynomial and all polynomials of degree ≤ 2. Find a basis of the subspace $U = \{p \in \mathbb{P}_2 | p(2) = 0\}$ of $\mathbb{P}_2$, and determine $\dim U$. (You have two pages to complete this problem.)

Answer: An arbitrary polynomial $p \in \mathbb{P}_2$ can be written in the form $p = a_0 + a_1 x + a_2 x^2$ for arbitrary $a_0, a_1, a_2 \in \mathbb{R}$. For such a polynomial $p(2) = a_0 + 2a_1 + 4a_2$. Hence $p(2) = 0 \iff a_0 + 2a_1 + 4a_2 = 0 \iff a_0 = -2a_1 - 4a_2$. Hence

$$\begin{aligned} U &= \{-(2a_1 + 4a_2) + a_1 x + a_2 x^2 \,|\, a_1, a_2 \in \mathbb{R}\} \\ &= \{a_1(-2 + x) + a_2(-4 + x^2) \,|\, a_1, a_2 \in \mathbb{R}\} \\ &= \mathit{span}\,\{x - 2, x^2 - 4\}. \end{aligned}$$

But $x - 2$ and $x^2 - 4$ are two polynomials of different degrees and are therefore linear independent (§5.2, Example 2; class of Nov. 30). Of course, this can also easily be checked directly: suppose $a_1(x-2) + a_2(x^2 - 4) = 0_{\mathbb{P}_2}$, i.e., $-2a_1 - 4a)2 + a_1 x + a_2 x^2$ is the zero polynomial, hence all coefficients are zero. In particular, looking at the coefficient of x and of x^2 we see that $a_1 = 0 = a_2$. Hence, $\{x - 2, x^2 - 4\}$ is a spanning set of U as well as linearly independent. Therefore $\{x - 2, x^2 - 4\}$ is a basis of U. The dimension of U is 2, since we have a basis with 2 elements.

Reference: suggested exercise §5.2, 3b; a similar problem was done in the problem session of Dec. 3.

Marking: 4 points for writing U as a span, 4 points for linear independence, 2 point for finding the dimension of U.

16. (4 bonus points) Let V be a vector space, and let $\mathbf{u}, \mathbf{v} \in V$. Use the definition of a subspace to prove that $\mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{v} = \{a\mathbf{u} + b\mathbf{v} : a, b \in \mathbb{R}\}$ is a subspace.

Answer: We abbreviate $U = \mathbb{R}\mathbf{u} + \mathbb{R}\mathbf{v}$ and check the 3 conditions defining a subspace.

(1) We have $0_V \in U$ since $0_V = 0\mathbf{u} + 0\mathbf{v}$, where $0 = 0_{\mathbb{R}} \in \mathbb{R}$.

(2) Suppose $x, y \in U$, say $x = a\mathbf{u} + b\mathbf{v}$ and $y = c\mathbf{u} + d\mathbf{v}$. Then $x + y = (a + c)\mathbf{u} + (b + d)\mathbf{v}$, and this is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, hence lies in U.

(3) Let $c \in \mathbb{R}$ and $x \in U$, say $x = a\mathbf{u} + b\mathbf{v}$. Then $cx = (ca)\mathbf{u} + (cb)\mathbf{u} \in U$. Thus, the 3 conditions defining a subspace hold for U.