How to get the Taylor series of $\ln(x)$ from the Taylor series of $\dfrac{1}{x}$

We know that the Taylor series for $\frac{1}{x}$ (centered at $c = 1$) is

$$\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n = 1 - (x-1) + (x-2)^2 - \cdots.$$

Given that $\int \frac{1}{x}dx = \ln(x) + C$, we integrate both sides of the equation above to obtain

$$\ln(x) + C = x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots. \tag{1}$$

Now we have to find the constant C. We substitute $x = 1$ in the equation that we have found

$$\ln(1) + C = 1 - \frac{(1-1)^2}{2} + \frac{(1-1)^3}{3} - \cdots$$

and use the fact that $\ln(1) = 0$ to deduce that $C = 1$.

Therefore Equation (1) can be written as

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots = \sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{n}$$

and this gives the Taylor series for $\ln(x)$ centered at $c = 1$.

Caution! The way this problem is done in the Solutions Guide is not quite right, as they do not justify how they found the constant C. Please write to me if you have any questions on this.