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1. (2 points) Determine if the vector $(1, 0)$ is a linear combination over $\mathbf{C}$ of the vectors $(1, 2i)$ and $(1, 1+i)$ in $\mathbf{C}^2$. If this is the case, give a presentation of $(1, 0)$ as a linear combination of these vectors.

$$(1,0) = a(1,2i) + b(1,1+i)$$
$$(1,0) = (a+b,\ 2ia + (1+i)b)$$
$$\Rightarrow \begin{cases} a+b=1 \longrightarrow b = 1-a \\ 2ia+(1+i)b=0 \longrightarrow 2ia + (1+i)(1-a) = 0 \end{cases}$$
$$\Rightarrow 2ia + 1 - a + i - ia = 0 \Longrightarrow (i-1)a + i+1 = 0 \Longrightarrow a = \frac{i+1}{1-i}$$
$$\Rightarrow b = 1-a = 1 - \frac{i+1}{1-i} = \frac{1-i-i-1}{1-i} = \frac{-2i}{1-i} = \frac{2i}{i-1}$$

So $(1,0)$ is a linear combination of $(1,2i)$, $(1,1+i)$ and

Answer: $\boxed{(1,0) = \frac{i+1}{1-i}(1,2i) + \frac{2i}{i-1}(1,1+i)}$

2. (3 points) Show that the functions $f, g, h\colon \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = 1,\ g(x) = x \text{ and } h(x) = \cos(x)$$

are linearly independent over $\mathbf{R}$.

Suppose for some $a, b, c \in \mathbb{R}$ we have

$$af(x) + bg(x) + ch(x) = 0$$

We have to show $a = b = c = 0$.

$$a + bx + c\cos(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

In particular:

1) $x = 0 \Rightarrow a + c = 0 \Longrightarrow a = -c$

So our equation is $-c + bx + c\cos(x) = 0$

2) $x = \pi/2 \Rightarrow -c + \frac{b\pi}{2} + 0 = 0 \Longrightarrow b = \frac{2c}{\pi}$

So our equation becomes $-c + \frac{2c}{\pi}x + c\cos(x) = 0$

3) $x = 3\pi/2 \Rightarrow -c + \frac{2c}{\pi} \cdot \frac{3\pi}{2} + 0 = 0$

$\Rightarrow -c + 3c = 0 \Longrightarrow 2c = 0 \Rightarrow c = 0$

(from 1) & 2))

$\Longrightarrow a = 0$ & $b = 0$. So f, g, h are linearly independant

3. (4 points) Recall that $\mathbf{P}_n(\mathbf{R})$ is the set of all polynomials of degree less than or equal to **n**. Let U be a subset of $\mathbf{P}_n(\mathbf{R})$ defined as

$$U = \{p(x) \in \mathbf{P}_n(\mathbf{R}) \mid p(1) = p(2)\}$$

(i) Show that U is a subspace of $\mathbf{P}_n(\mathbf{R})$.

Suppose $p(x), q(x) \in U$ and $a, b \in \mathbb{R}$. We have to show that $\underset{(ap+bq)(x)}{\underset{\shortparallel}{ap(x)+bq(x)}} \in U$

$(ap+bq)(1)$

$= ap(1)+bq(1)$

$= a\,p(2)+bq(2)$ because $p(1)=p(2)$ and $q(1)=q(2)$

$= (ap+bq)(2)$

So $(ap+bq)(x) \in U$ and hence U is a subspace of $P_n(R)$.

(ii) In the case where $n = 2$, find a basis for U.

If $n=2$, then we can write U as

$$U = \{ax^2+bx+c \mid a,b,c \in \mathbb{R},\ \overset{(a(1)^2+b(1)+c)}{a+b+c} \underset{(*)}{=} \overset{(a(2)^2+b(2)+c)}{4a+2b+c}\}$$

The equation $(*)$ comes down to

$$a+b = 4a+2b \Rightarrow 3a+b=0 \Rightarrow b=-3a$$

So U can be written as

$$U = \{ax^2-3ax+c \mid a,c \in \mathbb{R}\} = \{a(x^2-3x)+c \mid a,c \in \mathbb{R}\}$$

So every element of U can be written as a linear combination of x^2-3x and 1, which means that $\{x^2-3x, 1\}$ is a spanning set for U. But this set is also linearly independent,

Answer: $\{x^2-3x, 1\}$ is a basis.

↳ Since if $a(x^2-3x)+b.1 = 0 \Rightarrow ax^2-3ax+b=0 \Rightarrow a=b=0$

4. (5 points) Let $U = \{A \in M_{2,2}(\mathbb{R}) \mid A = A^t\}$ where A^t denotes the transpose of A.

(i) Determine a basis for U. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$. Then $A = A^t$,

which means $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Longrightarrow b = c$.

So $U = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \middle| a, b, d \in \mathbb{R} \right\}$

Clearly, every element of U can be written as

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = a \underset{\overset{\shortparallel}{A1}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + b \underset{\overset{\shortparallel}{A2}}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + d \underset{\overset{\shortparallel}{A3}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}, \quad (A1, A2, A3 \in U)$$

So $\{A1, A2, A3\}$ is a spanning set for U. Moreover, A1, A2, A3 are linearly independent, since

$$aA1 + bA2 + cA3 = 0 \Longrightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow a = b = c = 0$$

So $\{A1, A2, A3\}$ is a basis for U.

Answer: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(ii) Extend the basis that you found in part (i) to a basis of $M_{2,2}(\mathbb{R})$.

A standard basis for $M_{2,2}(\mathbb{R})$ is $\left\{ \underset{\overset{\shortparallel}{A1}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overset{\underset{\shortparallel}{B}}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{\underset{\shortparallel}{C}}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{\overset{\shortparallel}{A3}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \right\}$

If you add $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to the basis of U in part (i), you have a linearly independ set

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = \{A1, A2, A3, B\}$$

these are linearly independent since

$$aA1 + b \quad + cA3 + dB = 0 \Longrightarrow \begin{pmatrix} a & b+d \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow a = b = c = d = 0$$

We have a set of 4 linearly independent elements in a 4-dim'l space, so we have a basis

Answer: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

5. (6 points) Let V and W be vector spaces over a field K.

(i) Give the definition of a linear transformation T from V to W.

A linear transformation $T: V \longrightarrow W$ is a map that satisfies the following property:

$\forall u, v \in V$ and $a, b \in K$ $\quad T(au+bv) = aT(u) + bT(v)$.

(ii) Give the definition of the kernel $\ker(T)$ of a linear transformation $T: V \to W$.

$\text{Ker}(T) = \{ v \in V \mid T(v) = 0 \}$

(iii) Prove that the kernel of a linear transformation $T: V \to W$ is a subspace of V.

Let $u, v \in \text{Ker}(T) \implies T(u) = T(v) = 0$

Let $a, b \in K$

$T(au+bv) = aT(u) + bT(v) = 0$

So $au+bv \in \text{Ker}(T)$

So $\text{Ker}(T)$ is a subspace of V.

Note It is clear from the definition of $\text{Ker}(T)$ that $\text{Ker}(T)$ is a subset of V. Also, $\text{Ker}(T)$ always contains 0, since $T(0) = T(0.v) = 0T(v) = 0 \quad \forall v \in V$

So $\text{Ker}(T)$ is not empty.

6. (5 points) Let $A = \begin{pmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 1 & 1 \\ 2 & -2 & 0 & 2 \end{pmatrix}$ and let $T_A\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation associated to A given by $T_A(X) = AX$ for all column vectors $X \in \mathbb{R}^4$.

(i) Determine a basis for $\ker T_A$.

Ker (T_A) consists of all vectors $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ such that $AX = 0$.
We first row reduce A:

$$\begin{array}{c} R1 \\ R2 \\ R3 \end{array}\begin{pmatrix} 1 & -1 & 1 & 3 \\ -1 & 1 & 1 & 1 \\ 2 & -2 & 0 & 2 \end{pmatrix} \xrightarrow[R3 \to R3+2R2]{R2 \to R1+R2} \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix} \xrightarrow[R3 \to R3-R2]{R2 \to \frac{1}{2}R2} \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R1 \to R1-R2} \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A'.$$

The solution set of $AX=0$ is the same to that of $A'X=0$, which is the solution to the system of linear equations: ?

$\begin{cases} x - y + w = 0 \\ z + 2w = 0 \end{cases}$ $\Rightarrow$ If you take y and w as parameters, you have the solution set as $(y-w, y, -2w, w) = y(1,1,0,0) + w(-1,0,-2,1)$

Answer: $\{(1,1,0,0), (-1,0,-2,1)\}$

We then show that these two vectors are linearly independent:
$a(1,1,0,0) + b(-1,0,-2,1) = 0$
$\Rightarrow (a-b, a, -2b, b) = 0 \Rightarrow a = b = 0$.

(ii) Determine a basis for $\operatorname{Im} T_A$.

The image of T_A is the same as the column space of A. To find a basis for the column space of A, we note that the columns in A' that contain the pivot elements are Columns 1 & 3. So Columns 1 & 3 of the original matrix A form a basis for $\operatorname{Im} T_A$.

Note Alternatively, by row reducing the matrix whose rows are the columns of A, you can find a basis for the column space of A, or $\operatorname{Im} T_A$.

Answer: $\{(1,-1,2), (1,1,0)\}$