October 28^{th}, 2004

MAT 2341

Recall: • A linear map $F: U \to V$ U, V vector spaces over field K

where $u, v \in U$, $a, b \in K$

such that $F(au+bv) = aF(u) + bF(v)$

• $\ker F = \{u \in U \mid F(u) = 0\}$

• $\operatorname{Im} F = \{v \in V \mid \exists u \in U \; F(u) = v\}$

• $\dim \ker F + \dim \operatorname{Im} F = \dim U$

F is: injective (one-to-one)
: if $F(u) = F(v)$ implies that $u = v$

surjective (onto)
: if $\operatorname{Im} F = V$, or equivalently, $\forall v \in V \; \exists u \in U$ such that $F(u) = v$

bijective (one-to-one correspondence)
: if F is both surjective & injective

DEF'N: A bijective linear map is called an 'isomophism'

EX

$F : \mathbb{R}^2 \to \mathbb{R}^2$

$F(x) = Ax \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Is F surjective? injective? bijective?, Find a rule for F.

(b) $F((x,y)) = F((z,w)) \implies$ does $(x,y) = (z,w)$

$\hookrightarrow (x+2y, 2x+y) = (z+2w, 2z+w)$

$$\begin{cases} x + 2y = z + 2w \\ 2x + y = 2z + w \end{cases} \longrightarrow \begin{cases} 2x + 4y = 2z + 4w & ① \\ 2x + y = 2z + w & ② \end{cases}$$

① - ② $\implies 3y = 3w$

$\boxed{y = w}$

$\implies \boxed{x = z}$

$\therefore$ F is injective

DEF'N $F: U \to V$ map of vector spaces F is <u>non-singular</u> if $\ker F = \{0\}$

[This means that if $F(v) = 0$ then $v = 0$]

Otherwise, F is called "singular".

THEOREM

$F: U \to V$ map of k-vector spaces. Then F is injective iff F is non-singular.

* PROOF: 2 parts

1) F is injective $\Rightarrow$ F is nonsingular

Suppose $u \in \ker(F) \Rightarrow F(u) = 0 = F(0)$

↳ by def'n of a linear map

$\Rightarrow u = 0$, because F is injective

$\Rightarrow \ker F = \{0\}$, F is nonsingular

2) F is nonsingular $\Rightarrow$ F is injective

Suppose $u, v \in U$, $F(u) = F(v)$

$\Rightarrow F(u) - F(v) = 0$

$F(u-v) = 0$

$\therefore u - v \in \ker(F)$

$u - v = 0$

$u = v$

$\therefore F$ is injective

BACK TO EX

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad F((x,y)) = (x+2y, 2x+y)$$

To show F is injective, now it is enough to show F is nonsingular.

$$F((x,y)) = 0$$

$$\Rightarrow (x+2y, 2x+y) = (0,0)$$

$$\begin{cases} x+2y=0 \\ 2x+y=0 \end{cases} \Rightarrow \begin{cases} -2x-4y=0 & ① \\ 2x+y=0 & ② \end{cases}$$

$$① + ② \qquad -3y = 0 \Rightarrow \boxed{y=0}$$

$$x+2y=0 \quad \boxed{x=0}$$

$(x,y) = (0,0)$, F is nonsingular

THEOREM

$F: U \to V$ is a nonsingular linear map of k-vector spaces. Then the image of any linearly independent set of vectors in U is a linearly independent set of vectors in V.

PROOF: Suppose $\{u_1, \ldots, u_n\}$ is a set of lin. indep. vectors in U. Suppose $\exists a_1, \ldots, a_n \in K$ such that $a_1F(u_1) + a_2F(u_2) + \ldots + a_nF(u_n) = 0$

we want to show $a_1 = a_2 = \ldots = a_n = 0$

$\Rightarrow F(a_1u_1 + a_2u_2 + \ldots\ldots + a_nu_n) = 0$

$\Rightarrow a_1u_1 + \ldots\ldots + a_nu_n \in \ker(F)$ & F is nonsingular, $\therefore \ker(F) = \{0\}$

$\Rightarrow a_1u_1 + \ldots\ldots a_nu_n = 0$

$\Rightarrow a_1 = a_2 = \ldots\ldots = a_n = 0$

$\therefore F(u_1), F(u_2), \ldots, F(u_n)$ are linearly independent.

<u>In our EX</u> $F: \mathbb{R}^2 \to \mathbb{R}^2$, nonsingular

$F(x,y) = (x+2y, 2x+y)$

Take $\{e_1, e_2\} = \{(1,0), (0,1)\}$

By last Theorem, $F(e_1), F(e_2)$ are lin. indep.

$\therefore \underset{(1,2)}{F(e_1)}, \underset{(2,1)}{F(e_2)}$ make a basis for $\mathbb{R}^2$