MAT2341, 11/01

Recall:
If you have F:U->V a linear map, U,V vector spaces over a field k, then f is called nonsingular if kerF = {0}.
If F is nonsingular => to F is one to one.
F is an isomorphism if F is one-to-one and onto (linear map that is bijective)

Theorem
Suppose F:U->V is a linear map of vector spaces over K and dimU is finite & dimU=dimV, then F is an isomorphism iff F is nonsingular.

Proof:
1) Suppose F is nonsingular
=> From previous class, this means F is 1 to 1. So we need F onto.
Recall: dimU=dim(KerF)+dim(ImF)

$$\underset{\dim V}{\underset{\parallel}{\dim U}} = \underset{0}{\underset{\parallel}{\dim \operatorname{Ker} F}} + \dim \operatorname{Im} F = \dim \operatorname{Im} F$$

$$\Rightarrow \dim V = \dim \operatorname{Im} F, \quad \operatorname{Im} F \subset V \Rightarrow V = \operatorname{Im} F$$

Side note: $A \subseteq B$ vspaces
$\dim A \leq \dim B$
if $\dim A = \dim B \Rightarrow A = B$

$$\Rightarrow \left.\begin{array}{l}\text{F is onto}\\ \text{F is 1-1}\\ \text{F is linear}\end{array}\right\} \Rightarrow \text{F is an isomorphism.}$$

2) Suppose F is an isomorphism
=> F is 1-1 (by definition)
=> F is nonsingular. ∴

Operations on linear maps

Take any 2 linear maps
F:U->V
G:U->V
linear maps of K-vector spaces.
We can define:
F+G:U->V
(F+G)(U)=F(U)+G(U)
$k \in K$ kF:U->V (kF)(u) = kF(u)
Note: F+G, kF are linear maps.

> The above is equivalent to: If F,G $\in$ Hom(U,V) then F+G $\in$ Hom(U,V) and kF $\in$ Hom(U,V) $\forall$ k $\in$ K

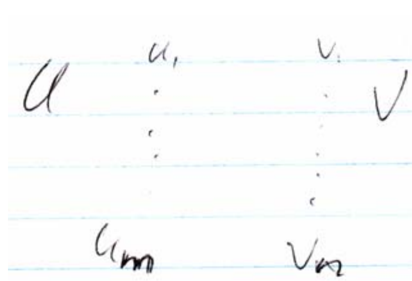
Pick u,v $\in$ U, a,b $\in$ K
(F+G)(au+bv) = [want to get a(F+G)(u) + b(F+G)(v)]
By definition this is = F(au+bv) + G(au+bv)
> Note: if confused, you can replace (F+G) by H if it helps

= aF(u) + bF(v) + aG(u) + bG(v)
= a(F(u)+G(u)) + b(F(v)+G(v)) = a(F+G)(u) + b(F+G)(v) $\therefore$

Note: kF is linear (Exercise!)

Definition:
For fixed k-vector spaces U and V, Hom(U,V) is the set of all linear maps from U to V.

Theorem:
Hom(U,V) as defined above is a vector space.
If dimU = m & dimV = n then dimHom(U,V) = mn.



Composition of linear maps

F:U->V, G:V-> F,G linear maps U,V,W k-vector spaces.

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[rr, bend right, "GF"'] & V \arrow[r, "G"] & W \end{tikzcd}$$

$$u \longrightarrow F(u) \longrightarrow G(F(u))$$

$$u \in U \quad (G \circ F)(u) = G(F(u))$$

<u>Fact</u>GoF is a linear map
u,v ∈ U; a,b ∈ K
(GoF)(au+bv) = a(GoF)(u) + b(GoF)(v)
because
(GoF)(au+bv) = G(F(au+bv))
Next step: G(F(au+bv)) = G(aF(u) + bF(v))

$$\underset{F\ \text{linear}}{=} G\Big(\overbrace{aF(u)}^{\in U} + \overbrace{bG(v)}^{\in V}\Big)$$

$=_{\text{(G linear)}}$ aG(F(u))+b(G(F(v))) = a(GoF)(u) + b(GoF)(v)

Ex: Let's map F,G:R^2->R^2
F(x,y) = (x+y,0)
G(x,y) = (-y,x)

<u>Find</u>
F+G
FoG

Sol: both F+G and FoG map to R^2.
(F+G)(x,y) = F(x,y) + G(x,y) = (x+y,0) + (-y,x) = (x+y-y,x) = (x,x). It IS a linear map because we have already proved that a linear map + another = a linear map.
(FoG)(x,y) = F(G(x,y)) = F((-y,x)) = (x-y,0)

Note: F,G are "linear operators"

<u>Definition:</u>
A <u>linear operator</u> on a vector space V is a linear map from V to V.
Note for those who are confused: (FoG)(U)=F(G(U))

In this setting, we can think the "product" of two linear operators F&G and define it as:
FG=FoG
GF=GoF.

It makes sense to talk about the inverse of a linear operator F:V->V.

<u>Definition</u>:
The <u>inverse</u> of a linear operator F:V->V is a linear map F^{-1} s.t. FoF^{-1} = idv and $F^{-1}oF$ = idv (idv is the identity map from V to V, that is that it takes an element U and gives out U, like multiplying something by 1).

Reminder: for F to be invertible, we need it to be bijective, that is, a map that is 1-to-1 and onto. Since F is linear, this means that for it to be invertible, it needs to be an isomorphism.

ex: $T:R^3 \to R^3$
T(x,y,z) = (2x,4x-y,2x+3y-z)
- Is T invertible? (by the next theorem, it's enough to show that T is nonsingular)
- If so, find T^{-1}.

i) Check if kerT = {0}
Suppose T(x,y,z) = 0
=> (2x,4x-y,2x+3y-z = 0) = (0,0,0) => 2x=0, 4x-y=0, 2x+3y-z=0 => (x,y,z)=0 => kerT={0} => Tnonsingular and T invertible

T(x,y,z)=(u,v,w) = (2x,4x-y,2x+3y-z)
find T^{-1}(u,v,w)=(x,y,z)

$$\begin{cases} u = 2x \Rightarrow x = u/2 \\ v = 4x - y \\ w = 2x + 3y - 2 \end{cases} \Rightarrow \begin{cases} x = u/2 \\ v = 4(u/2) - y \Rightarrow y = 2u - v \\ w = 2(u/2) + 3(2u - v) - 2 \\ \Rightarrow z = u + 6u - 3v - w = 7u - 3v - w \end{cases}$$

$T^{-1}:R^3 \to R^3$
T^{-1}(u,v,w)=(u/2, 2u-v, 7u-3v-w)

Suppose you have a map $T:R^2 \to R^2$ linear map. You can always write as a matrix.
Suppose E = $\{e_1,e_2\}$, $[T]_E$ = matrix of T with respect to basis E.
T(x,y) = (x+y), (2x-y)
$T(e_1)$ = T(1,0)=(1,2)
$T(e_2)$ = T(0,1)=(1,-1)

$[T]_E = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

$T(x,y)=[T]_E \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x-y \end{bmatrix}$

Theorem:

Suppose F is a linear operator on vector space V over a field k (F:V->V linear map) then the following are equivalent:

1) F is nonsingular (kerF = {0})
2) F is 1-to-1 (2 implies 3 because dimV = dimV, i.e. dim(Domain) = dim(Target))
3) F is onto (3 implies 4 because 1-to-1 + onto + linear is an isomorphism which means invertible)
4) F is invertible