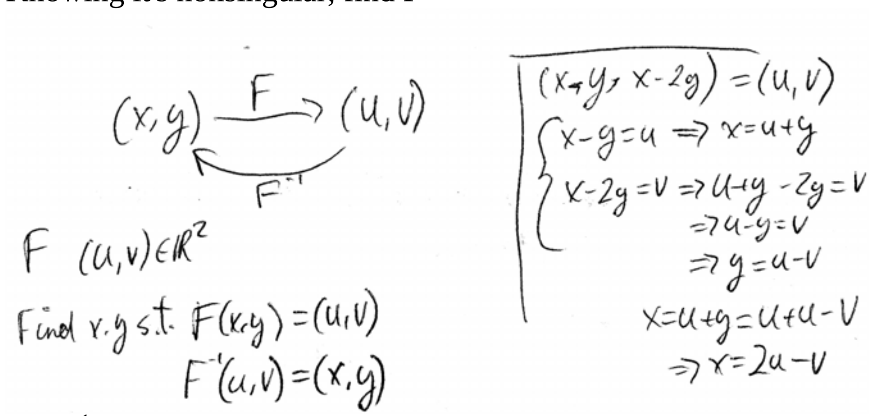
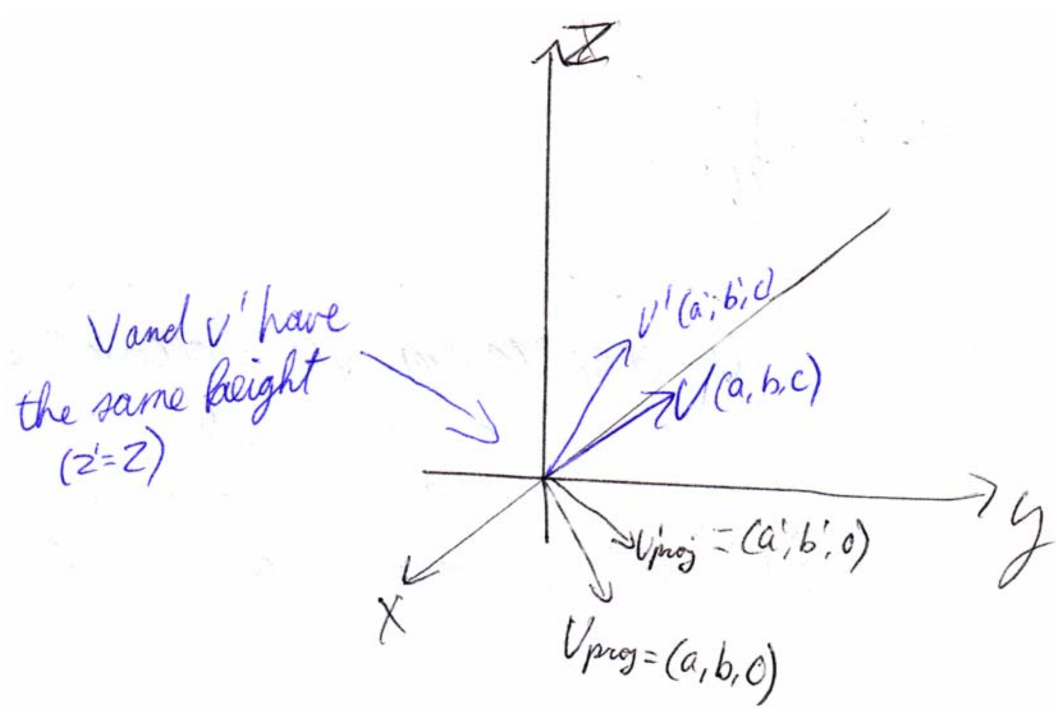
MAT2341, 11/04

Sidenote about the problems:
5.25: $F:R^2 \to R^2$ is $F(x,y)=(x-y, x-2y)$
Knowing it's nonsingular, find F^{-1}



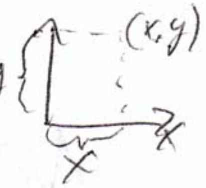
So, $F^{-1}(u,v) = (2u-v, u-v)$

Example 5.7 p. 177
Having $G:R^3 \to R^3$ rotation $v \in R^3$ about the z-axis through an angle Θ.
So $G(x,y,z)=(x\cos\Theta - y\sin\Theta, x\sin\Theta + y\cos\Theta, z)$

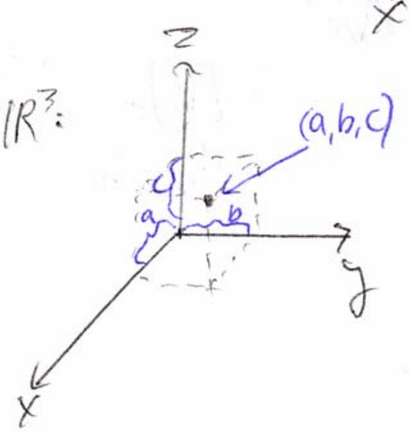


Sidenote:

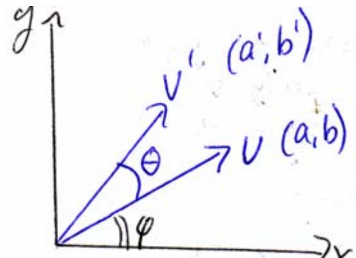
in $\mathbb{R}^2$



in $\mathbb{R}^3$:



In practice, what G does is leave the z coordinate as it is, and rotate the projection of v onto the xy-plane by Θ degrees.



Question reduces to how to describe v' in this setting.

Fix $|v| = d$
↳ length of v.

So $|v'| = d$.

Recall: $\begin{cases} a = d\cos\varphi \\ b = d\sin\varphi \end{cases}$

Question: $\begin{cases} a' = ? \\ b' = ? \end{cases}$

$$\begin{cases} a' = d\cos(\Theta+\varphi) = d(\cos\Theta\cos\varphi - \sin\Theta\sin\varphi) \\ b' = d\sin(\Theta+\varphi) = d(\sin\Theta\cos\varphi + \sin\varphi\cos\Theta) \end{cases}$$

$$\Rightarrow \begin{cases} a' = \overbrace{d\cos\varphi}^{a}\cos\Theta - \overbrace{d\sin\varphi}^{b}\sin\Theta \\ b' = \underbrace{d\cos\varphi}_{a}\sin\Theta + \underbrace{d\sin\varphi}_{b}\cos\Theta \end{cases}$$

(a,b)
↓ rot. by θ
$(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

$$\Rightarrow \begin{cases} a' = a\cos\Theta - b\sin\Theta \\ b' = a\sin\Theta + b\cos\Theta \end{cases}$$

So $G(x,y,z) = (x\cos\Theta - y\sin\Theta,\ x\sin\Theta + y\cos\Theta,\ z)$.

Matrices of linear operators

V k-vector space, T: V-->V linear operator.

Suppose S = $\{u_1...u_n\}$ basis for V
$\forall$ v $\in$ V, $\exists$ $a_1...a_n \in$ k s.t. v = $a_1u_1+...+a_nv_n$.

Ex: R^2
basis E = $\{e_1, e_2\}$ = {(1,0), (0,1)}
basis S = $\{u_1, u_2\}$ = {(1,1), (-1,0)}

<u>check basis</u> $au_1 + bu_2 = 0$
(a-b, a) = 0
=> a = 0 => b = 0
(5,7) $\in R^2$
E: (5,7) = 5(1,0) + 7(0,1) = $5e_1 + 7e_2$.
Now we want to write 5,7 with S: (5,7) = a(1,1) + b(-1,0) = $7u_1+7u_2$ = 7(1,1) + 2(-1,0)

We want $\begin{cases} a-b=5 \\ \boxed{a=7} \end{cases} \Rightarrow b=2$

So $[v]_E = \begin{bmatrix}5\\7\end{bmatrix}$, $[v]_S = \begin{bmatrix}7\\2\end{bmatrix}$

<u>Question</u> How to find $[T]_S$ which is the matrix describing the linear operator T in terms of the basis S?
$\forall$ v $\in$ V T: $[v]_S$ --> $[T]_S[v]_S$
(the resulting matrix is the coordinates of the image of v in terms of the basis S)
$[T]_S$ is determined as follows:
Find $[T(u_1)]_S,...,[T(u_n)]_S$

$$T(u_i) = b_1u_1 + b_2u_2 + ... + b_nu_n \Rightarrow [T(u_i)]_S = \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}$$

Then $[T]_S = [[T(u_1)]_S ... [T(u_n)]_S]$
(where the $[T(u_i)]_S$ are the columns of $[T]_S$).

Example: suppose T:R^2-->R^2
T(x,y) = (x + 2y, 3x + 5y)
E = $\{e_1, e_2\}$ S = $\{u_1, u_2\}$ = {(1,1), (-1,0)} be basis of R^2

<u>Q1</u>: Find $[T]_E$ T(1,0) = (1,3) T(0,1) = (2,5)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow \begin{matrix} a = 1 \\ c = 3 \end{matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} b = 2 \\ d = 5 \end{matrix}$$

$$\Rightarrow [T]_{\mathrm{E}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Claim

$T(e_1) = (1,3) \quad T(e_2) = (2,5)$

$$[T(e_1)]_{\mathrm{E}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \; [T(e_2)]_{\mathrm{E}} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$[T]_{\mathrm{E}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

Checking if $[T]_E$ works.

Take any (x,y) $\in$ R^2

That is: $[(x,y)]_E = \begin{bmatrix} x \\ y \end{bmatrix}$

(x,y) = $xe_1 + ye_2$

$$[T]_{\mathrm{E}}[(x,y)]_{\mathrm{E}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 3x+5y \end{bmatrix} = (x+2y)e_1 + (3x+5y)e_2$$

$$= (x+2y, 3x+y)$$

$$= T(x,y)$$

<u>Q2</u>: Find $[T]_S$

T:R^2-->R^2 is T(x,y) = (x + 2y, 3x + 5y)

S = {u_1, u_2} = {(1,1), (-1,0)}

$[T]_S = [[T(u_1)]_S, [T(u_2)]_S]$

$T(u_1)$ = T(1,1) = (3,8) = au_1+bu_2

$T(u_2)$ = T(-1,0) = (-1,3) = cu_1+du_2

u_1:

(3,8) = (a,a) + (-b,0) = (a-b, a)

$$\begin{cases} a-b=3 \\ a=8 \end{cases} \Rightarrow b = 5$$

=> $T(u_1) = 8u_1 + 5u_2$ => $[T(u_1)]_S = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

u_2: (-1,3) = (c-d, c)

$$\Rightarrow \begin{cases} c-d=1 \\ c=3 \end{cases} \Rightarrow d = -2 \Rightarrow [T(u_2)]_S = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

$T(u_2) = -3u_1 - 2u_2$

=> [T]s = $\begin{bmatrix} 8 & -3 \\ 5 & -2 \end{bmatrix}$

---> This means, that for any vector v $\in$ R^2, if you write v = au_1 +bu_2 (a,b unique because {u_1, u_2} is a basis)

then $[T(v)]_S = [T]_S \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8a-3b \\ 5a-2b \end{bmatrix}$

(note that the a and b above are not the same as a,b,c,d we started with)

T(v) = (8a-3b)u_1 + (5a+2b)u_2

<u>Ex</u>: Check T(1,3) is the same under applying T directly to (1,3), or doing $[T]_S[(1,3)]_S$

<u>Example</u>: V vector space of functions w/ basis $S = \{\sin t, \cos t, 3^{3t}\}$

D:V-->V

D(f(t)) = f'(t)

Find [D]s

$$D(\sin t) = \cos t = 0\sin t + 1\cos t + 0e^{3t} \rightarrow [D(\sin t)]_S = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$D(\cos t) = -\sin t = -1\sin t + 0\cos t + 0e^{3t} \rightarrow [D(\cos t)]_S = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(e^{3t}) = e^{3t} = 0\sin t + 0\cos t + 3e^{3t} \rightarrow [D(e^{3t})]_S = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$[D]_S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$