Nov. 8th 2004

Practice Problems

6.3 $\quad G: \mathbb{R}^2 \to \mathbb{R}^2 \qquad S \text{ basis} = \{u_1, u_2\} = \{(1,3),(2,5)\}$

$G(x,y) = (2x-7y,\ 4x+3y)$

(a) Find $[G]_S$

(b) Verify $[G]_S[v]_S = [G(v)]_S$

(a) $G(u_1) = G((1,3)) = (-19, 13)$

$= au_1 + bu_2 = a(1,3) + b(2,5) = (a+2b,\ 3a+5b)$

Find a, b $\quad \begin{cases} a+2b = -19 \\ 3a+5b = 13 \end{cases} \Rightarrow -b = -3(-19)+13 = 70 \qquad \boxed{b=-70}$

$a = -19-2b$

$= -19+140 = 121 \qquad \boxed{a=121}$

$$[G(u_1)]_S = \begin{bmatrix} 121 \\ -70 \end{bmatrix}$$

$G(u_2) = G(2,5) = (-31, 23) = cu_1 + du_2$

$c = 201 \qquad d = -116$

$$[G(u_2)]_S = \begin{bmatrix} 201 \\ -116 \end{bmatrix}$$

$$[G]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix}$$

THEOREM $\quad V$ vector space, T linear operator on V, S a basis of V, Then $\forall v \in V$

$$[T(v)]_S = \underbrace{[T]_S[v]_S}_{\text{PRODUCT OF 2 MATRICES.}}$$

(b) $G(v) = G(4,-3) = (8+21, 16-9) = (29,7) = au_1 + bu_2$

$(29,7) = (a+2b, 3a+5b)$

$\begin{cases} a+2b = 29 \\ 3a+5b = 7. \end{cases} \Rightarrow -b = -87+7 = -80 \Rightarrow \boxed{b = 80}$

$a+2b = 29 \Rightarrow 29-160 = -131 \ \boxed{a = -131}$

$$[G(v)]_S = \begin{bmatrix} -131 \\ 80 \end{bmatrix}$$

$[v]_S \quad (4,-3) = au_1 + bu_2 = (a+2b, 3a+5b)$

$\begin{cases} a+2b=4 \\ 3a+5b=-3 \end{cases} \Rightarrow -3a-6b=-12$

$\Rightarrow -b = -15 \ \boxed{b = 15}$

$a = 4-2b = -26$

$\boxed{a = -26}$

$$[v]_S = \begin{bmatrix} -26 \\ 15 \end{bmatrix}$$

$$[G]_S[v]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix} \begin{bmatrix} -26 \\ 15 \end{bmatrix} = \begin{bmatrix} -131 \\ 80 \end{bmatrix} = [G(v)]_S$$

Theorem

V n-dim'l vector space/k, S basis of V.
F, G linear operators on V. $k \in K$. Then we have

(i) $[\underbrace{F+G}_{\text{sum of linear maps}}]_S = \underbrace{[F]_S + [G]_S}_{\text{sum of matrices}}$

(ii) $[kF]_S = k[F]_S$

(iii) $[F \circ G]_S = [F]_S[G]_S \rightarrow$ PRODUCT OF MATRICES

Change of Basis

V vector space, with two bases

$S = \{u_1, \dots, u_n\} \longrightarrow$ "Old basés"
$S' = \{v_1, \dots, v_n\} \longrightarrow$ "New basés"

Looking for "The change of basis matrix from S to S'"

$v_1 = a_1^1 u_1 + a_2^1 u_2 + \dots + a_n^1 u_n \qquad [v_1]_S = \begin{bmatrix} a_1^1 \\ a_2^1 \\ \vdots \\ a_n^1 \end{bmatrix}$

$v_2 = a_1^2 u_1 + a_2^2 u_2 + \dots + a_n^2 u_n \qquad [v_2]_S$

$\vdots$

$v_n = a_1^n u_1 + \dots + a_n^n u_n \qquad [v_n]_S = \begin{bmatrix} a_1^n \\ a_2^n \\ \vdots \\ a_n^n \end{bmatrix}$

EX $\mathbb{R}^3$ $S = \{u_1, u_2, u_3\} = \{(1,2,0), (1,3,2), (0,1,3)\}$

$\mathcal{E} = \{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

(a) Find change of basis matrix $\mathcal{E} \to S$

$u_1 = ?e_1 + ?e_2 + ?e_3$

$u_1 = (1,2,0) = 1e_1 + 2e_2 + 0e_3$ $\quad u_2 = (1,3,2) = 1e_1 + 3e_2 + 2e_3$ $\quad u_3 = (0,1,3) = 0e_1 + 1e_2 + 3e_3$

$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

PROP: Change of basis matrix from the standard basis $\mathcal{E}$ of K^n to any other basis S of K^n is the matrix P whose columns are the transpose of the (basis) vectors in S

(b) $S \to \mathcal{E}$

$[e]_S$

e_1 $(1,0,0) = (a, 2a, 0) + (b, 3b, 2b) + (0, c, 3c)$
$= (7, -6, 4)$

e_2 $(0,1,0) = (-3, 3, -2)$

e_3 $(0,0,1) = (1, -1, 1)$

$$Q = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

FACT: $P^{-1} = Q$

THEOREM: If S & S' are two bases of the vector space V

P = change of basis matrix $S \to S'$
Q = " " " " $S' \to S$

Then P, Q are invertible matrices, and $P = Q^{-1}$
$P^{-1} = Q$

EX CHECK THIS!

HOW TO USE THIS

THEOREM P Change of basis matrix $S \to S'$
S, S' are bases of vector space V
Then $\forall v \in V$ we have:

$$P[v]_{S'} = [v]_S$$

$$[v]_{S'} = P^{-1}[v]_S$$

EX USING ABOVE EX

$v = (1,-1,5)$ $S' = \mathcal{E} = \{e_1, e_2, e_3\}$
$S = \{u_1, u_2, u_3\} = \{(1,2,0), (1,3,2), (0,1,3)\}$

$[v]_S$

$(1,-1,5) = au_1 + bu_2 + cu_3$
$= (a+b, 2a+3b+c, 2b+3c)$

$$\begin{cases} a+b=1 \\ 2a+3b+c=-1 \\ 2b+3c=5 \end{cases}$$

$\boxed{a = 15 \quad b = -14 \quad c = 11}$

$[v]_S = \begin{bmatrix} 15 \\ -14 \\ 11 \end{bmatrix}$ $[v]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$

CHECK

$$[v]_E = Q^{-1}[v]_S$$

$$\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 15 \\ -14 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix} \quad \checkmark$$

PRACTICE PROBLEM 6.8

$S = \{ e^{3t}, te^{3t}, t^2e^{3t} \}$ basis of a vector space V of functions $f: \mathbb{R} \to \mathbb{R}$

$D(f) = f'$ $\quad D: V \to V$

FIND $[D]_S$

$D(e^{3t}) = 3e^{3t}$ $\quad [D(e^{3t})]_S = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$D(te^{3t}) = e^{3t} + 3te^{3t}$ $\quad [D(te^{3t})]_S = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

$D(t^2e^{3t}) = 2te^{3t} + 3t^2e^{3t}$ $\quad [D(t^2e^{3t})] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$$\Rightarrow [D]_S = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

PRACTICE PROBLEMS: 6.7, 6.15, 6.17, 6.24, 6.26