MAT2341, 11/11

Recall: Suppose you have V a K-vectorspace and S, S' bases of V
P = change of basis matrix from S to S'.

Theorem: $\forall$ v $\in$ V, $[v]_{S'} = P^{-1}[v]_S$
$[v]_S=P[v]_{S'}$
Proof: Assume dimV=3 (proof in higher dimension is similar).
$S = \{u_1,u_2,u_3\}$
$S' = \{v_1,v_2,v_3\}$
$v_1 = a_1u_1+a_2u_2+a_3u_3$
$v_2 = b_1u_1+b_2u_2+b_3u_3$
$v_3 = c_1u_1+c_2u_2+c_3u_3$
where a_1,a_2,a_3 b_1,b_2,b_3 $c_1,c_2,c_3 \in$ K.

$$P=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Take v $\in$ V
$v = d_1v_1 + d_2v_2 + d_3v_3$ where $d_1,\ldots,d_3 \in$ K
$= d_1(a_1u_1+\ldots+a_3u_3) + d_2(b_1u_1+\ldots+b_3u_3) + d_3(c_1u_1+\ldots+c_3u_3)$
$= (d_1a_1+\ldots+d_3c_1)u_1 + (d_1a_2+\ldots+d_3c_2)u_2 + (d_1a_3+\ldots+d_3c_3)u_3$

$$[v]_{S'}=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$[v]_S=\begin{bmatrix} d_1a_1+d_2b_1+d_3c_1 \\ d_1a_2+d_2b_2+d_3c_2 \\ d_1a_3+d_2b_3+d_3c_3 \end{bmatrix}$$

Verify that $[v]_S=P[v]_{S'}$.

$$P=[v]_{S'}=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}=\begin{bmatrix} d_1a_1+d_2b_1+d_3c_1 \\ d_1a_2+d_2b_2+d_3c_2 \\ d_1a_3+d_2b_3+d_3c_3 \end{bmatrix}=[v]_S$$

Theorem: P is a change of basis matrix for vectorspace V and T is a linear operator on V. Then:

$[T]_{S'} = P^{-1}[T]_S P$

Proof: $[T]_S=P^{-1}[T]_S P$

Take any $v \in V$

$[T]_{S'}.[v]_{S'}=[T(v)]_{S'}$

$T: V \dashrightarrow V$

$v \dashrightarrow T(v)$

Let $u=T(v) \in V$, so

$[T]_{S'}.[v]_{S'}=[T(v)]_{S'}=[u]_{S'}$

$=P^{-1}[u]_S$

$=P^{-1}[T(v)]_S$

$=P^{-1}[T]_S[v]_S$

$=P^{-1}[T]_S P\,[v]_{S'}$ by previous theorem then

So I know $\forall\ v \in V$

$([T]_{S'})[v]_{S'}=(P^{-1}[T]_S P)[v]_{S'}$

$$\begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = [v]_{S'} \quad a_1 \dots a_n \in k$$

That means that, $\forall\ x \in K^n$

$[T]_{S'} X = P^{-1}[T]_S P X$

$\Rightarrow [T]_{S'} = P^{-1}[T]_S P$

Ex: $A=\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ $\quad A: R^2 \dashrightarrow R^2 \quad X \dashrightarrow AX$

Find the matrix representing A relative to the basis S={(1,3),(2,5)}

$[A]_S=?$

$T: V \to V$

S, S' bases of V

$$\underbrace{[T]_S \leftrightarrow [T]_{S'}}_{\text{matrices representing } T}$$

$[T]_{S'} = B'$

$T: V \to V$

$[V]_{S'} \to B[V]_{S'}$

P: change of basis E --> S

1) Find P

2) Have A

3) Find $[A]_S=P^{-1}AP$

1) $P=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$ (it's the transpose of the vectors)

2) done.

3) $P^{-1}=\begin{bmatrix}5 & 2\\ 3 & 1\end{bmatrix}$

$[A]_S=P^{-1}AP=\begin{bmatrix}-5 & 2\\ 3 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$

= find the answer.

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$\Rightarrow [T]_S$ & $[T]_{S'}$ are "similar" matrices.

Definition: Matrices A, B are similar if $\forall$ a matrix C s.t. $A=C^{-1}BC$.

Theorem: Two matrices represent the same linear operator if and only if they are similar (conjugates).

$V=R^3$

Fact any 3x3 matrix A gives a linear operator on R^3

T_A: R^3---> R^3

x----->Ax

Let B is any other 3x3 matrix when do A & B define the same linear operator?

If $\forall$ C s.t. $A=C^{-1}BC$ (i.e. A, B similar) This means that B represents the linear operator T_A under a change of basis determined by C.

$[T_A]_E=A$, and $\forall$ a basis of R^3 s.t. $[T_A]_S=B$ (and this is because A and B are similar matrices).

Example:

$V=R^2$, $A=\begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}$, $B=\begin{bmatrix}28 & 47\\ -15 & -25\end{bmatrix}$

Q: Do A and B describe the same linear operator on R^2?

A: Yes, because if $C=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$ then $B=C^{-1}AC$.

This means there is a linear operator $T:R^2$-->R^2 and $\forall$ S,S' basis of R^2 s.t. $[T]_S=A$ and $[T]_{S'}=B$

Matrices of General linear mappings

T: V----> V

$[T]_S$

$[T]_{S'}$

$[T]_S=P^{-1}[T]_{S'}P$

Can generalize all discussions from linear operators to general linear transformations

F:U---> V F linear transformation

U,V k-vector spaces

dimU=m, dimV=n

$S=\{u_1,...,u_m\}$ basis or U

$S'=\{v_1,...,v_n\}$ basis for V

$$F(u_1)=a_1^1v_1+a_2^1v_2+\ldots+a_n^1v_n\in V$$

$$F(u_2)=a_1^2v_1+a_2^2v_2+\ldots+a_n^2v_n\in V$$

$$\ldots$$

$$F(u_n)=a_1^nv_1+a_2^nv_2+\ldots+a_n^nv_n\in V$$

$$[F(u_1)]_{S'}=\begin{bmatrix}a_1^1\\ \ldots\\ a_n^1\end{bmatrix},\ [F(u_2)]_{S'}=\begin{bmatrix}a_1^2\\ \ldots\\ a_n^2\end{bmatrix},\ldots,[F(u_n)]_{S'}=\begin{bmatrix}a_1^n\\ \ldots\\ a_n^n\end{bmatrix}$$

$$=\begin{bmatrix}a_1^1 & a_1^2 & \ldots & a_1^m\\ \ldots & \ldots & \ldots & \ldots\\ a_n^1 & a_n^2 & \ldots & a_n^m\end{bmatrix}$$

$$U\rightarrow V$$

$$[u]_S\rightarrow[F]_{S,S'}[u]_S$$

<u>Define:</u>

$[F]_{S,S'}=[[F(u_1)]_{S'},[[F(u_2)]_{S'},...,[[F(u_n)]_{S'}]$

<u>Theorem</u>

F,U,V,S,S' as above

$\forall$ u $\in$ U

$[F(u)]_{S'}=[F]_{S,S'}[u]_S$

(note6)

<u>Recall:</u> U,V vector space / field k,

Hom(U,V) -->Vector space of all linear transformation from U to V.

Suppose dimU=m, dimV=n

<u>Theorem</u> Hom(U,V) is isomorphic to M_{nxm} {all nxm matrices}