$F: U \to V$

Thm F, U, V, S, S' as above

$\forall u \in U \quad [F(u)]_{S'} = [F]_{S,S'} [u]_S$

$$U \longrightarrow V$$

$$[u]_S \longrightarrow [F]_{S,S'} [u]_S$$

Recall ~~[illegible]~~ U, V vector spaces / k field

$Hom(U, V) \to$ vector space of all linear trans. from U to V

$\dim U = m$

$\dim V = n$

Then $Hom(U, V)$ is isomorphic to $M_{n \times m} = \{$all $n \times m$ matrices with entries in $k\}$

HW. Read all of section 6.5 (p. 212, 213)

Practice: 6.54, 6.31, 6.33, 6.64

— NOTES FOR MONDAY NOVEMBER 15 —

6.64 $H: \mathbb{R}^2 \to \mathbb{R}^2$

$H(x, y) = (2x + 7y, x - 3y)$

Bases of $\mathbb{R}^2$ $S = \{\overset{u_1}{(1,1)}, \overset{u_2}{(1,2)}\}$

$S' = \{\overset{v_1}{(1,4)}, \overset{v_2}{(1,5)}\}$

a) $[H]_{S,S'} \longrightarrow = \left[\underset{\begin{bmatrix} a \\ b \end{bmatrix}}{\overset{}{[H(u_1)]_{S'}}}, \underset{\begin{bmatrix} c \\ d \end{bmatrix}}{[H(u_2)]_{S'}}\right]$

b) $[H]_{S',S}$

a) $H(u_1) = H((1,1)) = (9,-2) = av_1 + bv_2$

$H(u_2) = H((1,2)) = (16,-5) = cv_1 + dv_2$

$(9,-2) = (a)(1,4) + (b)(1,5) = (a+b,\ 4a+5b)$

$-4\begin{cases} a+b = 9 \\ 4a+5b = -2 \end{cases} \Rightarrow \begin{matrix} b = -38 \\ a = 47 \end{matrix}$

$(16,-5) = (c+d,\ 4c+5d)$

$-4\begin{cases} c+d = 16 \\ 4c+5d = 5 \end{cases} \Rightarrow \begin{matrix} d = -69 \\ c = 16-d = 16+69 = 85 \end{matrix}$

$$[H]_{S,S'} = \begin{bmatrix} 47 & 85 \\ -38 & -69 \end{bmatrix}$$

b) $H(v_1) = H((1,4)) = (30,-11) = a\,u_1 + b\,u_2$

$H(v_2) = H((1,5)) = (37,-14) = c\,u_1 + d\,u_2$

Find a, b, c, d $\quad [H]_{S',S} = \cancel{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

c) $v = (0,1) \in \mathbb{R}^2$ find $H((0,1))$ using part (a)
(ie: using $[H]_{S,S'}$) $\underline{\text{thm } 6.10}$
↳ find $[H((0,1))]_{S'}$

In general:

$H(\underset{0e_1+1e_2}{(0,1)}) = (7,-3) = 7e_1 + -3e_2$

$H(x,y) = (2x+7y,\ x-3y) \quad [H]_E = \begin{bmatrix} 2 & 7 \\ 1 & -3 \end{bmatrix}$

$$(x,y) = xe_1 + ye_2 \Rightarrow [(x,y)]_{\mathcal{E}} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[H(x,y)]_{\mathcal{E}} = [H]_{\mathcal{E}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+7y \\ x-3y \end{bmatrix}_{\mathcal{E}}$$

$$H(x,y) = (2x+7y)e_1 + (x-3y)e_2$$

Now, in this particular case: $H: \mathbb{R}^2 \Rightarrow \mathbb{R}^2$
$S \qquad S'$

Take any $v \in \mathbb{R}^2$, $[H]_{S,S'} [v]_S = [H(v)]_{S'}$

$$v = (0,1) = a u_1 + b u_2 = a(1,1) + b(1,2)$$
$$(0,1) = (a+b, a+2b)$$

$$\begin{cases} a+b=0 \\ a+2b=1 \end{cases} \Rightarrow \begin{matrix} a=-b \\ -b+2b=1 \end{matrix} \Rightarrow b=1,\ a=-1$$

$$[(0,1)]_S = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$[H(0,1)]_{S'} = [H]_{S,S'} [(0,1)]_S = \overbrace{\begin{bmatrix} 47 & 85 \\ -38 & -69 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 38 \\ -31 \end{bmatrix}_{S'}}^{(*)}$$

Check! $H((0,1)) = (7,-3)$

$*$ means that
$$(7,-3) = 38 v_1 - 31 v_2$$
$$= 38(1,4) - 31(1,5)$$
$$= (7,-3) \checkmark$$

Chapter 9

Determinants

determinant : Square matrix $\longrightarrow$ scalar
$n \times n$

$n=2$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ ~~Ex: $\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix} = -1 - 0 = -1$~~

Ex: $\det\begin{pmatrix} 1 & 5 \\ 0 & -1 \end{pmatrix} = -1 - 0 = -1$

Another notation : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$n=3$ $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- Fix a row or column, say the first column
- Find det of matrix not containing a_{11}, a_{21}, a_{31}

$$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \overset{(-1)^{2+1}}{-} a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} \overset{(-1)^{3+1}}{+} a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{21})$$

Ex: $\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 3 \end{pmatrix} \overset{1^{st}\text{ Row}}{=} 0\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$= 0 - (3+1) + 3(1-0)$

$= -4 + 3 = -1$

$n \times n$ matrix $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{vmatrix}$

Pick any row or column (Say column 1)

$$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} + (-1)^{2+1} a_{21} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ a_{42} & \cdots & \cdots & a_{4n} \\ \vdots & & \ddots & \vdots \\ a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}$$

$$+ \cdots + (-1)^{n+1} a_{n1} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{(n-1)2} & \cdots & \cdots & a_{(n-1)n} \end{vmatrix}$$

Basic Properties:

1) $\det(AB) = \cancel{\det(A) * \det(B)}\ \det(A) \times \det(B)$
2) $\det(A^{-1}) = \det(A)^{-1} = \dfrac{1}{\det(A)}$

Characteristic Polynomials

$A_{n\times n}$ matrix with entries in K (field)

Recall $I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}$ $(A I_n = A)$

Def $\Delta_A(t) = \det(tI_n - A) = (-1)^n \det(A - tI_n)$
↳ characteristic polynomial of A

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad \Delta_A(t) = \det\left(t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\right)$

$$= \det\begin{pmatrix} t-1 & -2 \\ -3 & t-4 \end{pmatrix} = (t-1)(t-4) - 6$$

$$= t^2 - 5t + 4 - 6 \ \boxed{= t^2 - 5t - 2}$$

Thm. Any $n \times n$ matrix A is a root of its characteristic polynomial ie: $\Delta_A(A) = 0$

Check $A^2 - 5A - 2I$

$$= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \underbrace{\begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}$$

$$= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$= 0 \quad \checkmark$$

Def The roots of the characteristic polynomials of A are called the eigen values of A.

Ex: $A = \begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix}$

$$\Delta_A(t) = |tI_n - A| = \begin{vmatrix} t-3 & 4 \\ -2 & t+6 \end{vmatrix}$$

$$\cancel{= (t-3)(t+6) + 8 = t^2 - 9t} \qquad = (t-3)(t+6)$$

$$= t^2 + 3t - 18 + 8$$

$$= t^2 + 3t - 10$$

$$= (t+5)(t-2)$$

$\boxed{\text{eigen values } t = -5, \ t = 2}$

- if λ is an eigenvalue of A, then $v \in k^n$ is called an eigen vector of A (and λ) if $Av = \lambda v$

want to find eigen vectors.

$\lambda = 2$ Find $v \in \mathbb{R}^2$ such that $Av = \lambda v$

ie: $v = (a, b)$ such that $\begin{pmatrix} 3 & -4 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{cases} 3a - 4b = 2a \\ 2a - 6b = 2b \end{cases} \Rightarrow \text{Find } a, b$$