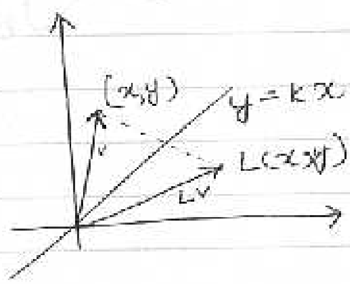
Problem

9.13

$L : \mathbb{R}^2 \to \mathbb{R}^2$ reflect^n across

line $y = kx$, $k > 0$



Note: Lv is not a ~~multp~~ multiple of v.

Show: a) $v_1 = (k, 1)$, $v_2 = (1, -k)$ } are eigenvectors of L.

b) show L is diagnolizable representation D of L.

Solut^n

→ normal vector to L is an eigenvector with $\lambda = -1$.

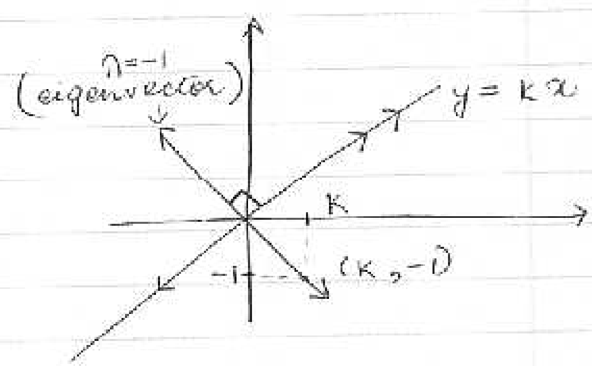
$\vec{(ax + by + cz)} = 0$ → plane passing through origin in $\mathbb{R}^3$.

normal vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Here we have :-

$kx - y = 0$

$v_1 = (k, -1)$



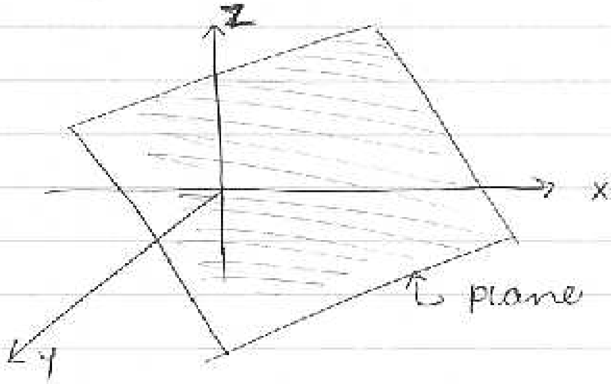
- any vector lying on the line $y = kx$, $\lambda_2 = 1$

$v_2 = (1, k)$

$D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$[L]_S = D$ where $S = \{v_1, v_2\} = \{(k,-1), (1,k)\}$
basis for $\mathbb{R}^2$

Ex. $L : \mathbb{R}^3 \to \mathbb{R}^3$ reflect^n across plane $2x + 3y - 4z = 0$
Find a diagonal representation of L if possible



$\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

2 possibilities for eigenspaces. (eigenvalues) :-

① span of normal vectors $\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$ corresponds to $\lambda = -1$.
dim = 1

② all vectors on the plane $2x+3y-4z=0$ " to $\lambda = 1$.
dim = 2.

→ [Vectors spanning the space, we already know, find the other.]

Chapter 7

Inner Product Spaces

note (→ Is a vector space, where we can do a certain kind of multiplicat^n ~~of~~ on every pair of vectors ~~space~~)

Let V be a real vector space (v. space / $\mathbb{R}$), $u, v \in V$. The inner product by $\langle u, v\rangle$ is a real value such that :

1) $a, b \in \mathbb{R}$, $u_1, u_2, v \in V$ $\langle au_1 + bu_2, v\rangle$
$= a\langle u_1, v\rangle + b\langle u_2, v\rangle$ → property (Linear)

Similarly,
$\langle v, au_1 + bu_2\rangle = a\langle v, u_1\rangle + b\langle v, u_2\rangle$

2) Symmetric $\langle u, v\rangle = \langle v, u\rangle$

3) Positive Definite $\langle u, u\rangle \geq 0$
& if $\langle u, u\rangle = 0$, then $u = 0$.

Ex. $-\mathbb{R}^n$ euclidean n-space, with inner product defined as " dot product ".
$u = (a_1, \dots a_n)$, $v = (b_1, \cdots b_n)$.
$u \cdot v = a_1b_1 + a_2b_2 + \cdots + a_nb_n$.

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Ex Check, when $n = 2$, that the dot product is an inner product
$\mathbb{R}^2$; $u = (a, b)$ $v = (c, d)$

— · —— · ——

Check: 1) Linear $u_1 = (a_1, b_1)$ $u_2 = (a_2, b_2)$
$v = (c, d)$ $k_1, k_2 \in \mathbb{R}^2$

now, $(k_1u_1 + k_2u_2) \cdot (v) = c(k_1a_1 + k_2a_2) + d(k_1b_1 + k_2b_2)$
$(k_1a_1 + k_2a_2, k_1b_1 + k_2b_2)$

Also, $k_1(u_1 \cdot v) + k_2(u_2 \cdot v) = k_1a_1c + k_1b_1d + k_2a_2c + k_2b_2d$ (3)
$= c(k_1a_1 + k_2a_2) + d(k_1b_1 + k_2b_2)$

2) Symmetric $u \circ v = v \circ u$?
$(ac + bd) = (ca + db)$ ← are equal.

3) Positive definite
$u = (a, b)$ $u \cdot u = a^2 + b^2 \geq 0$.
if $u \circ u = a^2 + b^2 = 0$ then $a = 0, b = 0$.

Ex $M_{m \times n} = \{$ all $m \times n$ matrices $\}$
↳ R-vector sp.
$A, B \in M_{m,n}$ $B_{m \times n} = B^T_{n \times m}$
$\langle A, B \rangle = tr(B^T A)$
↳ trace = sum of all diagonal entries of $B^T A$.

Ex $M_{1 \times 2}$ $A = [a, b]$ $B = [c, d]$ $\Rightarrow B^T = \begin{bmatrix} c \\ d \end{bmatrix}$

$B^T A = \begin{bmatrix} c \\ d \end{bmatrix}_{2 \times 1} [a, b]_{1 \times 2} = \begin{bmatrix} ca & cb \\ da & db \end{bmatrix}$

$\langle A, B \rangle = tr(B^T A) = ca + bd$
$\overset{\parallel}{\langle B, A \rangle}$

To check symmetric
$\langle B, A \rangle \rightsquigarrow A^T B = \begin{bmatrix} a \\ b \end{bmatrix} [c \ d] = \begin{bmatrix} ac & ad \\ bc & bd \end{bmatrix}$

$\langle B, A \rangle = tr(A^T B) = ac + bd = \langle A, B \rangle$

Check positive definite

$\langle A, A \rangle$ $a^2 + b^2 \geq 0$
if $\langle A, A \rangle = 0 \Rightarrow$ $a = 0$ & $b = 0$ $\Rightarrow A = [0, 0]$

Ex: $C[a,b] = \{$ all continuous fns. on the closed interval $[a,b]\}$.

Note: $C[a,b] \supseteq P(t)$ = all polynomials.

$f, g \in C[a,b] \quad \langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt.$

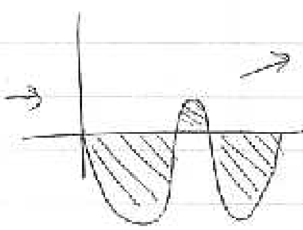
Ex: $C[0,1] \quad f(t) = t^2 \quad g(t) = t^3$

$\langle f, g\rangle = \int_0^1 t^2 \cdot t^3\, dt = \int_0^1 t^5\, dt = \left[\frac{t^6}{6}\right]_0^1 = \frac{1}{6}$

check $\langle f, f\rangle \geq 0.$

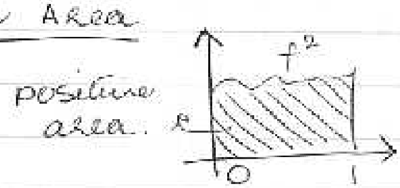
$\int_0^1 f^2(t)\, dt \geq 0.$

$\rightarrow$



negative integral.

Positive Area



Ex: Hilbert space $= \{$ all infinite sequences $(a_1, a_2, \ldots\ldots)$ such that $\sum_i (a_i)^2 = a_1^2 + a_2^2 + \cdots\cdots < \infty$ is convergent $\}$

$u = (a_1, a_2, \ldots)$, $v = (b_1, b_2, \ldots)$

$\langle u, v\rangle = u \cdot v = a_1b_1 + a_2b_2 + \cdots\cdots$ $\leftarrow$ defined

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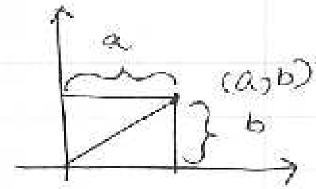
Definitn: Norm of a vector u in an Inner Product Space V.

$\|u\| = \sqrt{(u, u)} \leftarrow$ (norm/length of u)

(since $\langle u, u\rangle \geq 0$, $\sqrt{\langle u, u\rangle}$ makes sense)



Ex $u \in \mathbb{R}^2$ $u = (a, b)$

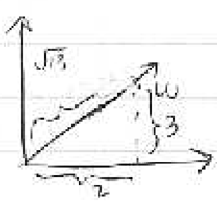
$\|u\| = \sqrt{a^2+b^2} = \sqrt{u \cdot u}$

- Normalizing of vector ~~[illegible]~~
(maintaining the direct^n while making the length equal to 1)

$$u \longrightarrow \frac{1}{\|u\|} u = \hat{u}$$

Ex $u = (2, 3) \in \mathbb{R}^2$

$\|u\| = \sqrt{2^2+3^2} = \sqrt{13}$, $\hat{u} = \frac{1}{\sqrt{13}}(2,3) = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$



Ex Matrix Space $M_{m,n}$.

$\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(A^T A)}$

$$\hat{A} = \frac{1}{\sqrt{tr(A^T A)}} A$$

Basic properties of $\|u\|$, $u \in V \rightarrow$ Inner Product sp.

(i) $\|u\| \geq 0$, if $\|u\| = 0$ then $u = 0$

(ii) $\forall k \in \mathbb{R}$ $\|ku\| = |k|\,\|u\|$

(iii) $\|u+v\| \leq \|u\| + \|v\|$ (triangle inequality