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Lecture NOV 24|11|04.

Last time:

→ Inner product space

↳ vector space + Inner product

K=R (over Real field)

$\forall\ u, v \in V \qquad \langle u, v\rangle \in R$

Norm of $u \in V$

$\|u\| = \sqrt{\langle u, u\rangle}$

- For $u \in R$ $\quad \|u\|$ = length of vector u $\quad \|u\| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$

Normalize $u \in V$

$\hat{u} = \frac{1}{\|u\|} u$

Prob 7.5

(i) $f(t) = t+2$, $g(t) = 3t-2$, $h(t) = t^2 - 2t - 3$.

$\langle f(t), g(t)\rangle = \int_0^1 f(t)\, g(t)\, dt$.

(a) Find $\langle f, g\rangle$

(b) Find norm of $\|g\|$

(c) Normalize g.

Sol^n(a) $\langle f, g\rangle = \int_0^1 f(t)\, h(t)\, dt = \int_0^1 (t+2)(t^2 - 2t - 3)\, dt$.

$= \int_0^1 (t^3 - 2t^2 - 3t + 2t^2 - 4t - 6)\, dt$

$= \int_0^1 \frac{t^4}{4} - 7 \times \frac{t^2}{2} - 6t\ ]$

$= \frac{1}{4} - \frac{7}{2} - 6 \quad \in R$

(b) $\|g\| = \sqrt{\langle g, g\rangle}$

$\langle g, g\rangle = \int_0^1 (3t-2)^2\, dt = \int_0^1 (9t^2 + 4 - 12t)\, dt$

$= \frac{9t^3}{3} - \frac{12t^2}{2} + 4t \Big|_0^1 = 3 - 6 + 4 = 1 \Rightarrow \|g\| = 1$.

(c) $\hat{g} = \frac{1}{\|g\|} \cdot g = g$ (since g is already normalized)

Theorem: (Cauchy Schwarz) V inner product space, $u, v \in V$ $|\langle u,v\rangle| \leq \|u\| \|v\|$ OR $\langle u,v\rangle^2 \leq \langle u,v\rangle\langle u,v\rangle$

Thm (Properties of Norm) V inner product space, $u, v \in V$

(a) $\|v\| \geq 0$, $\|v\| = 0$ if & only if $v = 0$.

(b) $\forall k \in \mathbb{R}$ $\|k \cdot v\| = \|k\| \cdot \|v\|$

(c) triangle inequality $\|u + v\| \leq \|u\| + \|v\|$

=> a) $\|v\| = \sqrt{\langle v, v\rangle} \geq 0$ (by def. of Inner Product & Norm)

suppose $\|v\| = 0$ => $\sqrt{\langle v,v\rangle} = 0$ => $v = 0$ (by def)

Now, suppose $v = 0$ => $\langle v, v\rangle = 0$ $\|v\| = 0$

=> b) $k \in \mathbb{R}$ $\|kv\| = \|k\| \|v\|$

$\|kv\| = \sqrt{\langle kv, kv\rangle} = \sqrt{k^2 \langle v, v\rangle} = |k| \sqrt{\langle v,v\rangle}$

$= |k| \cdot \|v\|$

=> c) P/o -> exercise

$\|u+v\| \leq \|u\| + \|v\|$

Ex

$\mathbb{R}^3$ $u = (1, 2, 3)$ $v = (0, 1, 0)$

$\|u\| + \|v\| \geq \|u + v\|$

$\|u\| = \sqrt{1+4+9} = \sqrt{14}$

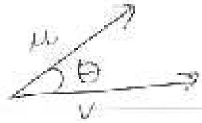
$\|v\| = \sqrt{1} = 1.$] + $\sqrt{14} + 1 = 4.74$

$\|u+v\| = \langle 1, 3, 3\rangle = \sqrt{1+9+9} = \sqrt{19} = 4.36$

∴ the prop. holds.

Angle between two vectors

V inner product space, the angle θ b/w 2 given vectors $u, v \in V$, is defined as an angle b/w $0 \leq \pi$ such that, $\cos\theta = \frac{\langle u, v\rangle}{\|u\| \|v\|}$



$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\cdot\|v\|}$ $\qquad \theta = \cos^{-1}\frac{\langle u, v\rangle}{\|u\|\cdot\|v\|} \quad 0 \leq \theta \leq \pi$

Ex Find angle b/w $u = (1,2,3)$, $v = (0,1,0)$

$\cos\theta = \frac{\langle u, v\rangle}{\|u\|\cdot\|v\|} = \frac{0+2+0}{\sqrt{14}} \Rightarrow \frac{2}{\sqrt{14}} < 1.$

Claim $-1 \leq \frac{\langle u, v\rangle}{\|u\|\,\|v\|} \leq 1$

$|\langle u, v\rangle| \leq \|u\|\,\|v\| \Rightarrow \left|\frac{\langle u, v\rangle}{\|u\|\cdot\|v\|}\right| \leq 1$ ← ratio of absolute values

Cauchy-Schwarz

$\therefore -1 \leq \frac{\langle u, v\rangle}{\|u\|\,\|v\|} \leq 1$ (Hence, proved)

Def: V inner product space, $u, v \in V$ are orthogonal if $\langle u, v\rangle = 0$

Note $\langle u, v\rangle = 0$, θ angle b/w $u, v \Rightarrow \cos\theta = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = 0$

$\Rightarrow \theta = \pi/2$

Ex $\langle (1,0), (0,1)\rangle = 0 + 0 = 0$ (basis) orthogonal

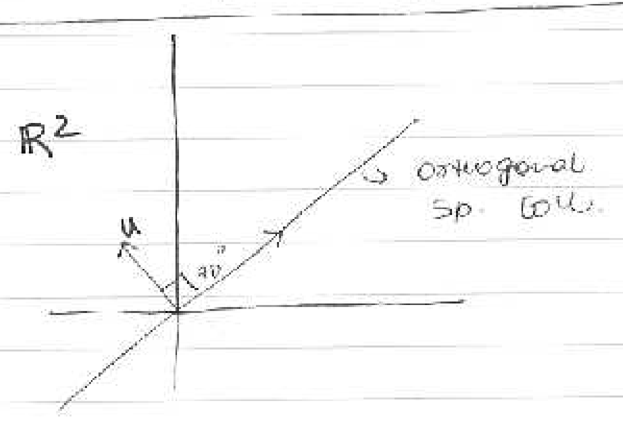
Ex $\langle (1,0), (1,1)\rangle = 1 + 0 = 1.$ (basis) not orthogonal

Ex $C[0,1] \rightarrow$ (cont. fn on $(0,\pi)$) (continuous functions on $[0,\pi]$)

Q sin and cos orthogonal? Yes.

$\langle \sin t, \cos t\rangle = \int_0^{\pi} \sin t \cos t \, dt$

$= \frac{1}{2}\sin^2 t \Big|_0^{\pi} = 0 - 0 = 0$



$\dim(u) + \dim u^{\perp} = 2$ (1)

$\Rightarrow \dim u^{\perp} = 1$

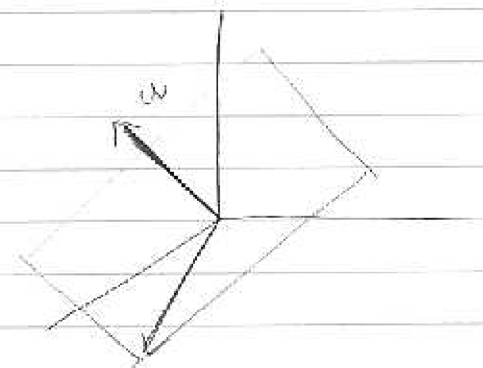
$u^{\perp} = \{ v \in \mathbb{R}^2 \mid \langle u, v \rangle = 0 \} = \{ (x,y) \in \mathbb{R}^2 \mid \langle (x,y), u \rangle = 0 \}$

Ex. $u = (-1, 2)$ $\quad u^{\perp} \quad \{ (x,y) \mid \langle (x,y), (1,2) \rangle = 0 \}$.

$u^{\perp} = \{ (x,y) \mid x + 2y = 0 \}$

$= \{ (-2y, y) \mid y \in \mathbb{R} \}$ → subspace of $\mathbb{R}^2$

Ex. $\mathbb{R}^3$



$\dim \langle u \rangle + \dim (u^{\perp}) = 3$, with $\dim \langle u \rangle = 1$

$\Rightarrow \dim u^{\perp} = 2$

Practice Problem. 7.6, 7.12.