(D)

7.12 $w = (1, 2, 3, 1)$ in R^4

$w^{\perp}$

$x + 2y + 3z + t = 0$

$v_1 = (0, 0, 1, -3)$

$w^{\perp} = \{(x, y, z, u) \in R^4 \mid (1, 2, 3, 1) \cdot (x, y, z, u) = 0\}$

$= \{(x, y, z, u) \mid x + 2y + 3z + u = 0$

apply at basis

$v = (1, 1, -1, 0) \in w^{\perp}$

find the basis for $w^{\perp}$

$w^{\perp}$ as before, find a basis

$w^{\perp}$ $\{x, y, z, u\} = u = x - 2y - 2z)$

depend

$= \{x, y, z, -x - 2y - 3z \mid x, y, z \in R\}$

free variable

$Basis = \{(1, 0, 0, -1), (0, 1, 0, -2), (0, 0, 1, -3)$

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Recall $u \in V$, V inner product space.

$$u^{\perp} = \{v \in V \mid \langle u, v \rangle = 0\}$$

$\{u\}^{\perp}$ a subset

$S \subseteq V$ is any subset of V,

then $S^{\perp} = \{v \in V \mid \langle u, v \rangle = 0 \ \forall u \in S\}$ complementary space to S

Fact $0 \in S^{\perp}$, for any subset S of V

Theorem. $S \subseteq V$, then $S^{\perp}$ is a subspace of V

read it for the exam

Pf $0 \in S^{\perp}$

any $a, b \in \mathbb{R}$, $u, v \in S^{\perp}$

$au + bv \in S^{\perp}$?

Let $w \in S$ $\langle au + bv, w \rangle$

because of linearity $= a\underbrace{\langle u, w \rangle}_{\substack{= 0 \\ \text{in } S^{\perp}}} + b\underbrace{\langle v, w \rangle}_{= 0}$

$= 0$

$\Rightarrow au + bv \in S^{\perp}$ ∎

Theorm

Let W be a subspace of an Inner product Space V. Then

$$V = W \oplus W^{\perp}$$

$\oplus$ ← direct sum.

ie $\forall v \in V$, $\exists$ unique $w \in W$ such that

$$v = w + w^{\perp}$$

Definition of direct sums:

also $W \cap W^{\perp} = \{0\}$

and $\dim V = \dim W + \dim W^{\perp}$

Example let $w = (1,2,3,1) \in \mathbb{R}^4$

W = subspace of $\mathbb{R}^4$ spanned by $w = (1,2,3,1)$

By theorm above

everything in W is perpendicular on $W^{\perp}$

$$\mathbb{R}^4 = W \oplus W^{\perp}$$

(4 find) sins 'ylo

$$= \text{span}(1,2,3,1) \oplus \text{span}\big((1,0,0,-1), (0,1,0,-2), (0,0,1,-3)\big)$$

4

① ③ $W^{\perp}$

Pick $V \in \mathbb{R}^4 \Rightarrow \exists$ unique a, b_1, b_2, b_3

4 find

$$V = \underbrace{a(1,2,3,1)}_{\in W} + \underbrace{b_1(1,0,0,-1) + b_2(0,1,0,-2) + b_3(0,0,1,-3)}_{\in W^{\perp}}$$

3

Def $S \subseteq V$ S is an orthogonal set if (60)
$\forall u, v \in S$ $\langle u, v\rangle = 0$ as long as $u \neq v$

Example $S = \{e_1, e_2, e_3\}$ in $\mathbb{R}^3$

$\langle e_1, e_2\rangle = \langle (1,0,0), (0,1,0)\rangle = 0+0+0 = 0$
$\langle e_1, e_3\rangle = \langle (1,0,0), (0,0,1)\rangle = 0 + 0 + 0 = 0$
$\langle e_2, e_3\rangle = \qquad = 0$

If additionally $\langle u, u\rangle = 1$ for any $u \in S$, then S is called orthonormal set

S is orthonormal.

$\langle e_1, e_1\rangle = 1+0+0 = 1$
$\langle e_2, e_2\rangle = 0+1+0 = 1$
lin ind orthogonal $\langle e_3, e_3\rangle = 0+0+1 = 1$

Theorm An orthogonal set of nonzero vectors is linearly independent

Pf (In case $|S| = 2$) $S = \{\overset{\neq 0}{u}, \overset{\neq 0}{v}\}$ orthogonal set
want to show that $a, b \in \mathbb{R}$ $au + bv = 0$
$\Rightarrow$ then $a = b = 0$

$\langle 0, u\rangle = 0$ $\forall u = V$
$\langle u, 0\rangle = 0$

$au+bv = 0$ $\Rightarrow \langle \underset{= 0}{au + bv}, u\rangle = 0$

$\langle u, u\rangle \neq 0$ if $u = 0$

$\langle u, u\rangle = \|u\|^2$ $c \neq 0$ $\|u\|^2$

(linearity) $\Rightarrow a\langle \underset{\neq 0}{u, u}\rangle + b\langle \underset{= 0}{v, u}\rangle = 0$
$0 \Rightarrow$ because S is an inner product so the orthogonal $= 0$.

$\Rightarrow a\langle u, u\rangle = 0$
$a = 0$

(4)

Similarly

$\langle au + bv , v \rangle = 0$

$\Rightarrow a \underbrace{\langle u, v \rangle}_{0} + b \langle v, v \rangle = 0$

inner product of each vector is nonzero

$b \underbrace{\langle v, v \rangle}_{\neq 0} = 0$

[illegible] the coefficient we want

$\boxed{b = 0}$

$\Rightarrow u, v$ linearly ind. ∎

Lemma

If $S = \{u_1, \dots, u_n\}$ is an orthogonal set

then $\|u_1 + u_2 + \dots + u_n\|^2 = \|u_1\|^2 + \|u_2\|^2 + \dots + \|u_n\|^2$

(5)

(61)

Example

$$S = \{\overset{u_1}{(1,1,1)}, \overset{u_2}{(1,-1,0)}, \overset{u_3}{(1,1,-2)}\}$$

Is S an orthogonal set?

$\langle (1,1,1), (1,-1,0) \rangle = 1 + (-1) + 0 = 0$

$\langle (1,1,1), (1,1,-2) \rangle = 1 + 1 - 2 = 0$

$\langle (1,-1,0), (1,1,-2) \rangle = \cancel{1} \cancel{-1} + 0 = 0$

$\langle (1,-1,0)(1,1,1) \rangle = \cancel{1} \cancel{-1} + 0 = 0$ $\Rightarrow$ S is orthogonal.

Is S an orthonormal? No

$\langle u_1, u_1 \rangle = 1 + 1 + 1 = 3$

Definition A basis S of an Inner prod space V is called an orthogonal basis if S is an orthogonal set

In our example above

S = is orthogonal

$S \Rightarrow$ lin ind

$|S| = 3 = \dim R^3 = 3 \Rightarrow S$ basis of R^3

enough elements to span it is li-d dim 3

find orthogonal basis

Example continue

Find $[(x,y,z)]_S$ $(x,y,z) \in R^3$

Find a, b, c such that

$(x,y,z) = au_1 + bu_2 + cu_3$

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$$\left[x, y, z \right]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$(x, y, z) = a(1,1,1) + b(1,-1,0) + c(1,1,-2)$$

$$\begin{cases} x = a + b + c \\ y = a - b + c \\ z = a - 2c \end{cases} \Rightarrow \text{Find } a, b, c \text{ in terms of } x, y, z \ldots$$

$$(x, y, z) = (a+b+c,\ a-b+c,\ a-2c)$$

Alternatively, you can use inner products:

R^3 Inner product space

S orthogonal basis

So we can use a different trick.

<u>want</u> a, b, c such that $v = x, y, z$

$v = au_1 + bu_2 + bu_3$ $\quad$ u_1 is orthogonal to u_2, u_3 [illegible]

$$\langle v, u_1 \rangle = \langle au_1 + bu_2 + cu_3, u_1 \rangle$$
$$= a\langle u_1, u_1 \rangle + b\cancel{\langle u_2, u_1 \rangle} + c\cancel{\langle u_3, u_1 \rangle}$$
$$= a\langle u_1, u_1 \rangle = 3a$$

similarly $\langle v, u_2 \rangle = b\langle u_2, u_2 \rangle = 2b$ $\qquad v = x, y, z$

$$\langle v, u_3 \rangle = c\langle u_3, u_3 \rangle = 6c$$

$$\langle v, u_1 \rangle = 3a \Rightarrow a = \frac{\langle v, u_1 \rangle}{3} = \frac{x+y+z}{3}$$

$$\langle v, u_2 \rangle = 2b \Rightarrow b = \frac{\langle v, u_2 \rangle}{2} = \frac{x-y}{2}$$

$$\langle v, u_3 \rangle = 6c \Rightarrow c = \frac{\langle v, u_3 \rangle}{6} = \frac{x+y-2z}{6}$$

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6c

$$a = \frac{x+y+z}{3}$$

$$b = \frac{x-y}{2}$$

$$c = \frac{x+y-2z}{6}$$

$$[x, y, z]_S = \begin{bmatrix} \frac{x+y+z}{3} \\ \frac{x-y}{2} \\ \frac{x+y-2z}{6} \end{bmatrix}$$

Thrm V Inn prod sp, S orthogonal basis $S = \{u_1, u_2, \ldots, u_n\}$

Then every $v \in V$ can be written as (uniquely)

$$v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 + \cdots + \frac{\langle v, u_n \rangle}{\langle u_n, u_n \rangle} u_n$$

$\Rightarrow$ Fourier Coefficients

ie

$$[v]_S = \begin{bmatrix} \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} \\ \frac{v, u_2}{u_2, u_2} \\ \frac{v, u_3}{u_3\, u_3} \\ \vdots \\ \frac{v\, u_n}{u_n\, u_n} \end{bmatrix}$$

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7.13 in the book.

(a) to show S is an orthogonal basis of R^4

$$S = \{ \overset{u_1}{(1,1,0,-1)}, \overset{u_2}{(1,2,1,3)}, \overset{u_3}{(1,1,-9,2)}, \overset{u_4}{(16,-13,1,3)} \}$$

$\langle u_1, u_2 \rangle = 1 + \cancel{2} - \cancel{3} = 0$

$\langle u_1, u_3 \rangle = \cancel{1} + 1 - \cancel{2} = 0$

$\langle u_1, u_4 \rangle = 16 - 13 - \cancel{3} = 0$

$\langle u_2, u_3 \rangle = 1 + 2 - 9 + \cancel{6} = 0$

$\langle u_2, u_4 \rangle =$

$\langle u_3, u_4 \rangle =$

$\Rightarrow$ S is an orthogonal set $\Rightarrow$ S lin ind

$\Rightarrow |S| = 4$ dim $R^4 \Rightarrow$ S basis

(b) proof