practic pro

7

7.13 in the book.

(a) to show S is an orthogonal basis of $\mathbb{R}^4$

$$S = \{ \overset{u_1}{(1,1,0,-1)}, \overset{u_2}{(1,2,1,3)}, \overset{u_3}{(1,1,-9,2)}, \overset{u_4}{(16,-13,1,3)} \}$$

$\langle u_1, u_2 \rangle = 1 + 2 - 3 = 0$

$\langle u_1, u_3 \rangle = 1 + 1 - 2 = 0$

$\langle u_1, u_4 \rangle = 16 - 13 - 3 = 0$

$\langle u_2, u_3 \rangle = 1 + 2 - 9 + 6 = 0$

$\langle u_2, u_4 \rangle =$

$\langle u_3, u_4 \rangle =$ $\Rightarrow S$ orthogonal set

~~which is the same dim = 4~~ $\Rightarrow S$ lind it has four 4 elts

$\Rightarrow S$ is an orthogonal set $\Rightarrow S$ lind ind

$\Rightarrow |S| = 4$ dim $\mathbb{R}^4 \Rightarrow S$ basis

(b) $[v]_S$ ie Find the coordinates of a given vector $v = (a,b,c,d)$ relative to basis S

(2) $[v]_S = \begin{bmatrix} \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} \\ \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} \\ \frac{\langle v, u_3 \rangle}{\langle u_3, u_3 \rangle} \\ \frac{\langle v, u_4 \rangle}{\langle u_4, u_4 \rangle} \end{bmatrix}$ Recall Fourier Coeffi!

(1,1)
(1) (1,0)
not orthogonal

Note if $\{u_1, \ldots, u_4\}$ were an orthonormal basis

ie if $\|u_i\| = 1 \Rightarrow \langle u_i, u_i \rangle = 1 \Rightarrow$ The Fourier Coefficients would not have had denominators]

(2)

(63)

$\langle v, u_1 \rangle = a + b - d$

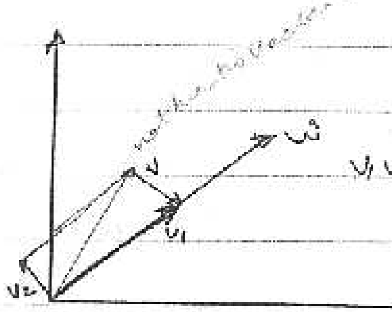
$\langle u_1, u_1 \rangle = 1 + 1 + 0 + 1 = 3$

$\langle v, u_2 \rangle = a + 2b + c + 3d$

$\langle u_2, u_2 \rangle = 1 + 4 + 1 + 9 = 15$

$$[y]_S = \begin{bmatrix} \frac{a+b-d}{3} \\ \frac{a+2b+c+3d}{15} \end{bmatrix}$$

orthogonal projection



v, w span $\mathbb{R}^2$

$v = v_1 + v_2$

$v_1 = \underbrace{cw}_{\text{multiple}} \quad c \in \mathbb{R}$

$v_2 = v - v_1 = v - cw$ orthogonal to w

$\Rightarrow \langle w, v - cw \rangle = 0$

$\langle w, v \rangle - c \langle w, w \rangle = 0$

(by linearity)

$$\boxed{\Rightarrow c = \frac{\langle w, v \rangle}{\langle w, w \rangle}}$$

$\langle w \rangle$

$w^{\perp}$ = ortho space to w

$\mathbb{R}^2 = \underbrace{\langle w \rangle}_{\text{span}} + w^{\perp}$

$\forall u \in \mathbb{R}^2$

$\exists u_1, u_2 \cdot u_1 \in \langle w \rangle$

$u_2 \in w^{\perp}$

$u = u_1 + u_2$

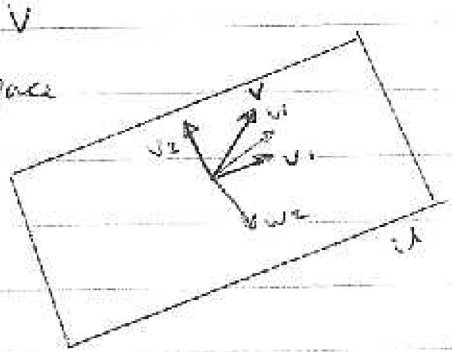
So if w is any vector $\neq 0$

v any other vector: $\in \mathbb{R}$ is $\boxed{c}$ w.

$= \text{proj}(v, w) = \boxed{\frac{\langle v, w \rangle}{\langle w, w \rangle}} w$

(3)

$U \subset V$
subspace



$S = \{w_1, w_2\}$
orthogonal basis for U

$V = U \oplus U^\perp$

$v \in V$ $\quad v = v_1 + v_2 \quad v_1 \in U \Rightarrow v_1 = c_1 w_1 + c_2 w_2$; $c_1, c_2 \in \mathbb{R}$

$v_2 = U^\perp \Rightarrow$ perpendicular $v_2 = v - v_1 = v - c_1 w_1 - c_2 w_2$

is orthogonal to w_1, w_2

Similar to before we can find c_1, c_2:

$$c_1 = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle}, \quad c_2 = \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle}$$

So $\text{proj}(v, U) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$

& $v - \text{proj}(v, U)$ is orthogonal to U

Thm Let V any inn prod sp. $\{w_1, \ldots, w_n\}$ be a set of orthogonal vectors. Then if $v \in V$ is any given vector:

$$\text{proj}(v, \text{span}(w_1, \ldots, w_n)) = c_1 w_1 + c_2 w_2 + \ldots + c_n w_n$$

where $c_i = \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle}$

orthogonal to

and $v - c_1 w_1 - c_2 w_2 - \ldots - c_n w_n$
is orthogonal to $w_1, w_2, \ldots, w_n$

④

(6a

<u>Use</u> this to build an orthogonal basis for V out of any given basis $\{v_1, v_2, \ldots v_n\}$

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$w_1 = v_1$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$ (the part of v_2 that is orthogonal to w_1)

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$$ [the part of v_3 orthogonal to w_1, w_2]

by scalar

$$w_n = v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \cdots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}$$

(The part of v_n orthogonal to $w_1, \ldots, w_{n-1}$)

Conclusion.

$\Longrightarrow$ $\{w_1, \ldots w_n\}$ is an orthogonal basis for V.

(5)

Example

U spanned by

$u_1 = (1,1,1,1)$ $u_2 = (1,1,2,4)$ $u_3 = (1,2,-4,-3)$

Find:

(a) orthogonal basis for U

(b) orthonormal " " U

$\boxed{w_1 = u_1 = (1,1,1,1)}$

$$w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \Rightarrow (1,1,2,4) - \frac{8}{4}(1,1,1,1)$$

$$\Rightarrow \boxed{w_2 = (-1,-1,0,2)}$$

$$w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle} \cdot w_2$$

$$= (1,2,-4,-3) - \frac{-4}{4}(1,1,1,1) - \frac{-1-2-6}{1+1+4}(-1,-1,0,2)$$

$$= (1,2,-4,-3) + (1,1,1,1) + \left(-\frac{3}{2}, -\frac{3}{2}, 0, 3\right)$$

$$\boxed{w_3 = \frac{1}{2}, \frac{3}{2}, -3, 1}$$

* Can replace w_3 by $2w_3$ which looks like $(1,3,-6,2)$

$\Rightarrow \{w_1, w_2, w_3\}$ orthogonal basis

(b) Orthonormal basis $= \left\{ \frac{1}{\|w_1\|} w_1, \frac{1}{\|w_2\|} \cdot w_2, \frac{1}{\|w_3\|} \cdot w_3 \right\}$

(6)

65

$\|W_1\| = \sqrt{4} = 2$

$\|W_2\| = \sqrt{6}$

$\|W'_3\| = \sqrt{1+9+36+4} = \sqrt{50}$

$$= \left\{ \frac{1}{2}(1,1,1,1),\ \frac{1}{\sqrt{16}}(-1,-1,0,2),\ \frac{1}{\sqrt{50}}(1,3,-6,2) \right\}$$

7-71

Example $U = M_{22}$, $\langle A,B\rangle = tr(B^t A)$

Find an orthogonal basis for the orthogonal complement of $\underbrace{\text{diagonal matrices}}_{U}$

Want orthogonal basis for $U^\perp$

$$U = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right\} \ \Big|\ a, b \in \mathbb{R} \quad = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Basis $U = S = \left\{ \underset{\substack{\| \\ U_1}}{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \underset{\substack{\| \\ U_2}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$

$\Rightarrow \dim U = 2$

$M_{22} = U \oplus U^\perp \Rightarrow \underset{\substack{\| \\ 4}}{\dim 22} = \underset{\substack{\| \\ 2}}{\dim U} + \dim U^\perp$ $\Downarrow_2$ has to be 2

$\underset{\substack{\| \\ U_3}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{\substack{\| \\ U_4}}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} \notin U$

Q/ $U_3, U_4 \in U^\perp$ [if so, they form a basis for $U^\perp$ if not, we use Fourier Coeff's to "take out" orthogonal parts"]

7

<u>Check</u>

$\langle u_3, u_1 \rangle = 0$?

$\langle u_3, u_2 \rangle = 0$?

$\langle u_4, u_1 \rangle = 0$?

$\langle u_4, u_2 \rangle = 0$?

$$\langle u_3, u_1 \rangle = tr(u_1^t, u_3) = tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= tr\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 0 \checkmark$$

$$\langle u_3, u_2 \rangle = tr(u_2^t, u_3) = tr\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\Rightarrow u_3 \in U^\perp$

$$\langle u_4, u_1 \rangle = tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \checkmark$$

$$\langle u_4, u_2 \rangle = tr\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = tr\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \checkmark$$

$\Rightarrow u_4 \in U^\perp$

$\Rightarrow u_3, u_4$ are two lin ind elements in $U^\perp$

$\Rightarrow \{u_3, u_4\}$ is a basis for $U^\perp$

To check if <u>orthogonal</u> basis

$$\langle u_4, u_3 \rangle = tr\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = tr\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$\Rightarrow \{u_3, u_4\}$ orthogonal basis for $U^\perp$.