Math 2341, Solutions to Homework No. 1

1. Let $f : X \to V$, $g : X \to V$ and $h : X \to V$ be functions from X to V, and let $a, b \in \mathbf{K}$. Now $\mathcal{O} : X \to V$ is given by $\mathcal{O}(x) = 0$ for all $x \in X$. For all $x \in X$, we have

$$\begin{aligned}(f + (g+h))(x) &= f(x) + (g+h)(x) = f(x) + (g(x) + h(x)) \\ &= (f(x) + g(x)) + h(x) = (f+g)(x) + h(x) \\ &= ((f+g)+h)(x),\end{aligned}$$

$(f + \mathcal{O})(x) = f(x) + \mathcal{O}(x) = f(x) + 0 = f(x)$,

$(a(bf))(x) = a(bf)(x) = a(bf(x)) = (ab)f(x) = ((ab)f)(x)$.

This shows that $f + (g+h) = (f+g)+h$, $f + \mathcal{O} = f$ and $a(bf) = (ab)f$.

2.

(a) Let $C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. We show that for every two $A, B \in U$ and $a, b \in \mathbf{R}$, $aA + bB \in U$. This means that we need to show that $(aA + bB)C = C(aA + bB)$.

By basic matrix operations, we have $(aA + bB)C = aAC + bBC)$. But since $A, B \in U$, we know that $AC = CA$ and $BC = CB$, so it follows that $(aA + bB)C = aCA + bCB = C(aA) + C(bB) = C(aA + bB)$, and U is a subspace of V.

(b) Here also U is a subspace of V. Again, pick $f, g \in U$ and $a, b \in \mathbf{R}$. We want to show that $(af + bg)''(t) + 2t(af + bg)'(t) + t(af + bg)(t) = 0$ for all $t \in (0, 1)$.

To see this, we note that:

$$\begin{aligned}&(af + bg)''(t) + 2t(af + bg)'(t) + t(af + bg)(t) \\ &= af''(t) + bg''(t) + 2t(af'(t) + bg'(t)) + t(af(t) + bg(t)) \\ &= af''(t) + 2atf'(t) + atf(t) + bg''(t) + 2btg'(t) + btg(t) \\ &= a(f''(t) + 2tf'(t) + tf(t)) + b(g''(t) + 2tg'(t) + tg(t)) \\ &= 0 + 0 = 0\end{aligned}$$

which proves that $af + bg$ is in U.

3. We look for a, b and c such that $M = aA + bB + cC$. This means that

$$a\begin{pmatrix} 1+i & 2i \\ 1 & -1+i \end{pmatrix} + b\begin{pmatrix} 1 & -1+i \\ 2i & 2-i \end{pmatrix} + c\begin{pmatrix} 1-i & -i \\ -2 & 1+i \end{pmatrix} = \begin{pmatrix} -1+i & -6-i \\ -5+5i & 3+3i \end{pmatrix}$$

which translates into the following system of linear equations:

$$\left\{\begin{array}{rcrcrcl} (1+i)a & + & b & + & (1-i)c & = & -1+i \\ 2ia & + & (-1+i)b & - & ic & = & -i \\ a & + & 2ib & - & 2c & = & -5+5i \\ (-1+i)a & + & (2-i)b & + & (1+i)c & = & 3+3i \end{array}\right.$$

The solution to this system is (you can write it as a matrix and use row reduction) $a = 1 + 2i$, $b = 2i$ and $c = 1 - 2i$, and hence:

$$M = (1+2i)A + 2iB + (1-2i)C.$$

4. The matrix for this system is

$$\begin{pmatrix} 2 & -4 & -1 & -1 \\ 1 & -2 & 1 & 4 \\ -2 & 4 & -1 & 5 \end{pmatrix}$$

which after row reductions becomes

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so our system of equations is equivalent to the following system:

$$\begin{cases} x - 2y + w = 0 \\ z + 3w = 0 \end{cases}$$

If we let $y = s$ and $w = t$, we find that $x = 2s - t$ and $z = -3t$. So the general solution of this system in $\mathbf{R}^4$ is

$$(x, y, z, w) = (2s - t, s, -3t, t) = s(2, 1, 0, 0) + t(-1, 0, -3, 1) \quad \text{for } s, t \in \mathbf{R}.$$

Hence $\{(2, 1, 0, 0), (-1, 0, -3, 1)\}$ is a generating set for the solution set in $\mathbf{R}^4$.

5. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to U, then for some $x, y, z \in \mathbf{R}$, we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = x \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} + y \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$$

which translates into the following system of linear equations:

$$\begin{cases} x + y + z = a \\ 2x - y = b \\ x + 2y - 2z = c \\ x - 3y + 2z = d \end{cases}$$

We write the corresponding matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 2 & -1 & 0 & b \\ 1 & 2 & -2 & c \\ 1 & -3 & 2 & d \end{array}\right)$$

which after row reductions becomes:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & -3 & -a + c \\ 0 & 0 & -11 & -5a + b + 3c \\ 0 & 0 & 0 & -b + c + d \end{array}\right)$$

So the system admits an equation if and only if $-b+c+d = 0$. In other words, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to U if and only if $-b + c + d = 0$.

6. We write a relation of linear dependence for f, g and h. Let $a, b, c \in \mathbf{R}$, and suppose

$$af + bg + ch = 0.$$

This means that

$$a(1/t) + b(1/(t - 1)) + c(1/(t^2 - t)) = 0 \quad \text{for all } t \in (0, 1).$$

The common denominator of the three fractions is $t^2 - t = t(t - 1)$, so we have (since t is never equal to 0 or 1, $t(t - 1)$ is not zero, so we can cancel the denominators):

$$\begin{aligned} & at(t - 1)/t + bt(t - 1)/(1 - t) + ct(t - 1)/(t^2 - t) = 0 && \text{for all } t \in (0, 1) \\ \Longrightarrow\ & a(t - 1) - bt + c = 0 && \text{for all } t \in (0, 1) \\ \Longrightarrow\ & (a + b)t + (c - a) = 0 && \text{for all } t \in (0, 1) \end{aligned}$$

So $a + b = 0$ and $c - a = 0$, which means that $b = -a$ and $c = a$. So for example, if $a = 1$, we can set $b = -1$ and $c = 1$, which means that we have the following relation of linear dependence between f, g and h (as well as many more, by setting different values for a):

$$f - g + h = 0.$$

Therefore the functions f, g and h are linearly dependent.