MATH 2341 - SOLUTIONS TO HW #2
FALL 2004

1. An element g of V can be written as

$$g = af_1 + bf_2 + cf_3$$

for $a, b, c \in \mathbb{R}$.

For g to be in U, we must have

$$g(0) = g(1) \iff af_1(0) + bf_2(0) + cf_3(0) = af_1(1) + bf_2(1) + cf_3(1)$$
$$\iff a + 2b = \tfrac{1}{2}a + b + c$$
$$\iff \tfrac{1}{2}a + b = c$$

So any element g of U can be written as

$$g = af_1 + bf_2 + (\tfrac{1}{2}a + b)f_3$$
$$= a(f_1 + \tfrac{1}{2}f_3) + b(f_2 + f_3)$$

for some $a, b \in \mathbb{R}$. In other words:

$$U = \{ a(f_1 + \tfrac{1}{2}f_3) + b(f_2 + f_3) \mid a, b \in \mathbb{R} \}$$

Therefore $\{ f_1 + \tfrac{1}{2}f_3, \ f_2 + f_3 \}$ is a generating set (or spanning set) for U.

To see if this is a basis, we check for linear independence.

If

$$a(f_1 + \tfrac{1}{2}f_3) + b(f_2 + f_3) = 0 \quad a, b \in \mathbb{R}$$

we have

$$\frac{a}{x+1} + \frac{a}{2}x^2 + bx^2 + b(2-x) = 0$$

To see what values a, b can have, we try assigning different values to x.

$x=0$ $\quad a+2b=0 \Rightarrow a+2b=0 \Rightarrow a=-2b$

$x=2$ $\quad \frac{a}{3}+\frac{a}{2}\cdot 4^2+b\,4=0 \Rightarrow \frac{7a}{3}+4b=0 \Rightarrow a=-\frac{12b}{7}$

$$\Rightarrow -2b=-\frac{12b}{7} \Rightarrow b=0 \Rightarrow a=0$$

So $f_1+\frac{1}{2}f_3$ and f_2+f_3 are linearly independend, and hence make a basis for U,

*[Note that, V is 3-dimensional (why?). And U is not equal to V (why?) so U has to have dimension strictly smaller than dimension of V, On the other hand, U contains two linearly independent elements, so it has dimension at least 2, Therefore dim $U=2$].

* You don't need this discussion the way the problem is written now.

2. W is the solution set associated to the matrix $(i, 1-i, -1)$ which has rank 1. From Theorem 4.19 it follows that the dimension of the solution space is $3-1=2$. So we need 2 elements in a basis for W, therefore (i) is not an option. Between (ii) and (iii), to check which one is a basis, we have to see which one satisfies the equation $ix+(1-i)y-z=0$ and is a linearly independent set. In (iii), $(0,1-i,1+i)\notin W$, because it does not satisfy the equation. So the only possible choice is (ii).

We check

$(1,1,1)$ $\quad i + (1-i) - 1 = 0$

$(i,1,-i)$ $\quad \underbrace{i^2}_{=-1} + 1 - i + i = 0$

So these are both solutions

To see if they are linearly independent, we check

$$a(1,1,1) + b(i,-1,-i) = 0$$

$$\Rightarrow \begin{cases} a + ib = 0 \\ a - b = 0 \\ a - ib = 0 \end{cases} \begin{matrix} \Rightarrow a = -ib \\ \Rightarrow a = b \\ \Rightarrow a = ib \end{matrix} \quad \Longrightarrow a = b = 0.$$

So (ii) is a basis

3) (i) We apply row reduction to A, and we have

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So rank $A = 3$. The dimension of the solution space is

$$\# \text{ unknowns} - \text{rank} A = 4 - 3 = 1.$$

The solution set is the same as the solution set of the row reduced matrix above, and hence the system of equations is

$$\begin{cases} x + z = 0 \\ y + z = 0 \\ w = 0 \end{cases} \Longrightarrow \begin{cases} x = -z \\ y = -z \\ w = 0 \end{cases}$$

So it is enough to fix z, and x, y will be determined by that. In other words, every solution to this

System is of the form

$$\begin{pmatrix} -z \\ -z \\ z \\ 0 \end{pmatrix} = z \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for any } z \in \mathbb{R}.$$

So $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for the solution set of $AX = 0$.

(ii) We have already reduced A to echelon form above, and the three rows are linearly independent, so they constitute a basis for the row space of A; i.e. the basis is $\{(1,0,1,0), (0,1,1,0), (0,0,0,1)\}$.

For the column space, notice that the columns in the echelon matrix that contain the pivots are columns 1, 2 and 4, so a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

4) - Any element of W_1 can be written as

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = x \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ generate W_1.

We show these three matrices are also linearly independent: If for some $x, y, z \in \mathbb{R}$

$$x\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + z\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

then

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which means that $x = y = z = 0$.

So they are lin. independent, and therefore

$$\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for W_1.

- A similar argument to the one for W_1 shows that

$$\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for W_2.

- Any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $W_1 \cap W_2$ must have the property that $b = -a$ (like all matrices in W_1)
and $c = -a$ (" " " " W_2)

So

$$W_1 \cap W_2 = \left\{ \begin{pmatrix} a & -a \\ -a & b \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$$

and like above, one can show that

$$\left\{ \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

is a basis for $W_1 \cap W_2$

- Since W_1 is generated by $\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and W_2 is generated by $\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $W_1 + W_2$ is generated by the <u>union</u> of these two sets, so a generating set for $W_1 + W_2$ is

$$\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

Now, the dimension of $W_1 + W_2$ is at most 4 (since $W_1 + W_2$ is a subspace of $M_{2\times 2}(\mathbb{R})$, which has dimension 4. Also note that the last 4 vectors in the set above are linearly independent. This is because if for some $x, y, z, u \in \mathbb{R}$ we have

$$x \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

then

$$\begin{pmatrix} z & u \\ x-z & y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Longrightarrow \begin{cases} z = 0 \\ u = 0 \\ x - 0 = 0 \Rightarrow x = 0 \\ y = 0 \end{cases}$$

So

$$\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is a basis for $W_1 + W_2$.

We verify the equation:

$$\underset{\substack{\| \\ 2}}{\dim(W_1 \cap W_2)} + \underset{\substack{\| \\ 4}}{\dim(W_1 + W_2)} = \underset{\substack{\| \\ 3}}{\dim W_1} + \underset{\substack{\| \\ 3}}{\dim W_2}$$

$$6 = 6 \quad \checkmark$$

5) (i) T is not linear, because

$$T(1,0) = (0,1,0)$$
$$T(0,1) = (0,0,1)$$

but $T((1,0)+(0,1)) = T(1,1) = (1,1,1)$

$$T(1,0) + T(0,1) = (0,1,1) \quad \times$$

(ii) T is linear. To see this this pick $p(x), q(x) \in P_2(\mathbb{R})$ and $a, b \in \mathbb{R}$. Then

$$T((ap+bq)(x)) = x(ap+bq)(x+1) = x\left(ap(x+1) + bq(x+1)\right)$$
$$= axp(x+1) + bx\,q(x+1)$$
$$= aT(p(x)) + bT(q(x))$$

6) We first find $\operatorname{Ker} T$.

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Ker} T$, then $\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} a-2c & b-2d \\ -2a+4c & -2b+4d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So $\begin{cases} a-2c=0 & (1) \\ b-2d=0 & (2) \\ -2a+4c=0 & (3) \\ -2b+4d=0 & (4) \end{cases} \Longrightarrow \begin{cases} a-2c=0 & \\ b-2d=0 & \\ 0=0 & (3)+2(1) \\ 0=0 & (4)+2(2) \end{cases}$

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$\Rightarrow \begin{cases} a=2c \\ b=2d \end{cases}$

So $\operatorname{Ker} T = \left\{ \begin{pmatrix} 2c & 2d \\ c & d \end{pmatrix} \middle| \ c, d \in \mathbb{R} \right\}$

Now for $c, d \in \mathbb{R}$

$$\begin{pmatrix} 2c & 2d \\ c & d \end{pmatrix} = c \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$$

And so $\operatorname{Ker} T$ is spanned by $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$.

Moreover, $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}$ are linearly independent, because:

$$a \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2a & 2b \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow a = b = 0$$

So $\left\{ \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \right\}$ is a basis for $\ker T$.

To find the image, we fix a basis of $M_{2,2}(\mathbb{R})$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We know the images (under T) of these basis elements span $\operatorname{Im} T$. So $\operatorname{Im} T$ is spanned by

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\| \qquad\qquad \| \qquad\qquad \| \qquad\qquad \|$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \quad A_3 = \begin{pmatrix} -2 & 0 \\ 4 & 0 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & -2 \\ 0 & 4 \end{pmatrix}$$

But $A_3 = -2A_1$ and $A_4 = -2A_2$, so we only need consider A_1, A_2 for a basis. Since A_1 and A_2 are linearly independent, they constitute a basis for $\operatorname{Im} T$.

linear independence:

$$a A_1 + b A_2 = 0 \Rightarrow \begin{pmatrix} a & b \\ -2a & -2b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow a = b = 0.$$

So $\operatorname{Im} T$ has $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \right\}$ as a basis.

$\boxtimes$