Solutions to HW3

1. $(1,2),(2,3)$ are linearly independent, as:

$$a(1,2)+b(2,3)=0 \Longrightarrow \begin{cases} a+2b=0 \\ 2a+3b=0 \end{cases} \Longrightarrow b=0 \Rightarrow a=0.$$

$\dim \mathbb{R}^2 = 2$, and we have two linearly independent vectors, so they must form a basis $\{(1,2),(2,3)\}$.

Now by Theorem 5.2, the map that takes $(1,2)$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $(2,3)$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ must be unique, so T is the unique such map.

(i) We first have to write a given point $(x,y) \in \mathbb{R}^2$ as a linear combination of $(1,2)$ and $(2,3)$.

$$(x,y) = a(1,2)+b(2,3) \qquad (\text{Find } a, b \in \mathbb{R})$$

$$\Longrightarrow (x,y) = (a+2b, 2a+3b)$$

$$\Rightarrow \begin{cases} x = a+2b \\ y = 2a+3b \end{cases} \Longrightarrow \boxed{2x-y=b} \Rightarrow x = a+2(2x-y)$$

$$\Longrightarrow a = x-4x+2y \Longrightarrow \boxed{a=2y-3x}$$

So, $(x,y) = (2y-3x)(1,2)+(2x-y)(2,3)$

Now, T is a linear map, so

$$T(x,y) = (2y-3x)T(1,2) + (2x-y)T(2,3)$$

$$= (2y-3x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + (2x-y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2x-y & -2y+3x \\ 2y-3x & 2x-y \end{pmatrix}$$

(ii) $T(1,1) = \begin{pmatrix} 2-1 & -2+3 \\ 2-3 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

2. (i) We find $\ker T$. Pick an element $p(x) = ax^2+bx+c \in P_2(\mathbb{R})$, and suppose $p(x) \in \ker T$. Then

$$T(p(x)) = 0 \Rightarrow (p(-1), p(0), p(1)) = (0,0,0)$$

$$\Rightarrow (a-b+c, c, a+b+c) = (0,0,0)$$

$$\Rightarrow \boxed{c=0}, \quad a-b=0, \; a+b=0 \Rightarrow 2a=0 \Rightarrow \boxed{a=0} \xRightarrow{a-b=0} \boxed{b=0}$$

$$\Rightarrow \ker T = \{0\} \quad (\text{or } T \text{ is non-singular})$$

$$\Rightarrow \dim \ker T = 0 .$$

We know that $\underbrace{\dim \mathbb{P}_2(\mathbb{R})}_{=3} = \underbrace{\dim \ker(T)}_{=0} + \dim \operatorname{Im}(T)$

$$\Rightarrow \dim \operatorname{Im}(T) = 3 = \dim \mathbb{R}^3 \Rightarrow T \text{ is surjective}$$

T is injective & surjective $\Rightarrow$ T is bijective

To find T^{-1}, suppose we are given an element $(u,v,w) \in \mathbb{R}^3$, and we want to find a $p(x) = ax^2+bx+c \in \mathbb{P}_2(\mathbb{R})$ such that

$$T(p(x)) = (u,v,w)$$

$$\Rightarrow (a-b+c, c, a+b+c) = (u,v,w)$$

$$\Rightarrow \begin{cases} a-b+c = u \\ c = v \\ a+b+c = w \end{cases} \Longrightarrow \begin{cases} 2b = w-u \Rightarrow \boxed{b = \frac{w}{2} - \frac{u}{2}} \\ \boxed{v=c} \\ a+b+c = w \Rightarrow a = w - \left(\frac{w}{2}-\frac{u}{2}\right) - v \\ \qquad \Rightarrow \boxed{a = \frac{w}{2} + \frac{u}{2} - v} \end{cases}$$

So $T^{-1}(u,v,w) = \left(\frac{w}{2}+\frac{u}{2}-v\right)x^2 + \left(\frac{w}{2}-\frac{u}{2}\right)x + v$.

[Check: $T\left(\left(\frac{w}{2}+\frac{u}{2}-v\right)x^2 + \left(\frac{w}{2}-\frac{u}{2}\right)x + v\right) = \left(\cancel{\frac{w}{2}} + \frac{u}{2} - \cancel{v} - \left(\cancel{\frac{w}{2}} - \frac{u}{2}\right) + \cancel{v}, v, \frac{w}{2} + \cancel{\frac{u}{2}} - \cancel{v} + \frac{w}{2} - \cancel{\frac{u}{2}} + \cancel{v}\right)$
$= (u, v, w)$ ✓]

2.(ii) We find $\operatorname{Ker}T$. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{Ker}T$.

$$\Rightarrow T(A) = 0 \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 2a & b+c \\ b+c & 2d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} 2a = 0 \\ b+c = 0 \\ 2d = 0 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = -c \\ d = 0 \end{cases}$$

$$\Rightarrow A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So a basis for $\operatorname{Ker}T$ consists of only one element $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$.

$\Rightarrow \dim \operatorname{Ker}T = 1$

and $\underbrace{\dim M_{2,2}(\mathbb{R})}_{=4} = \underbrace{\dim \operatorname{Ker}T}_{=1} + \dim \operatorname{Im}T$

$\Rightarrow \dim \operatorname{Im}(T) = 3$

T is singular, and hence does not have an inverse.

3. (i) $(2T - S)(p(x)) = 2T(p(x)) - S(p(x))$
$= 2(x+1)p(x) - p(2x+1)$

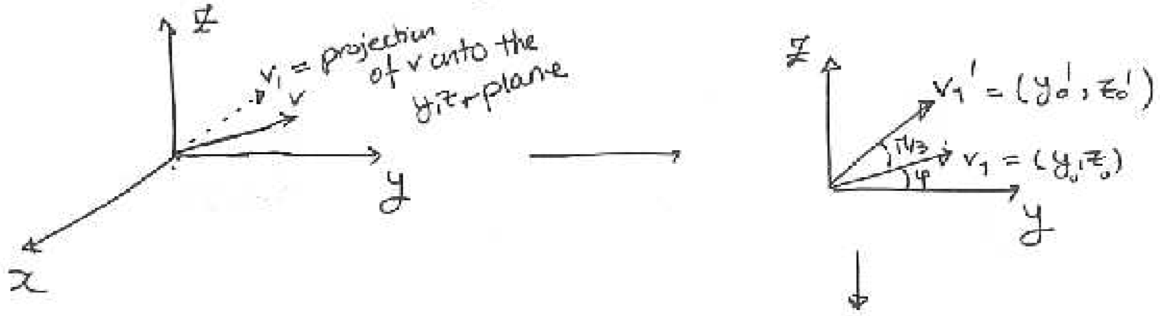
(ii) $(T \circ S)(p(x)) = T(S(p(x))) = T(p(2x+1))^{*}$
$= (x+1)\, p(2x+1)$

Hint: (*)
Think of $p(2x+1)$ as $q(x)$ (i.e., label it as a new polynomial, and then apply $T(q(x)) = (x+1)q(x)$
$= (x+1)p(2x+1)$

(iii) $(S \circ T)(p(x)) = S((x+1)p(x))$
$= (2x+1+1)p(2x+1) = (2x+2)p(2x+1)$

(iv) $(T - D \circ T)(p(x)) = T(p(x)) - D\circ T(p(x)) = (x+1)p(x) - D((x+1)p(x))$
$= (x+1)p(x) - [(x+1)p(x)]' = (x+1)p(x) - [p(x) + (x+1)p'(x)]$
$= (x+1)p(x) - p(x) + (x+1)p'(x) = xp(x) - (x+1)p'(x)$.

4. Since R is rotation about the x-axis, it leaves the x-coordinate of a given point in $\mathbb{R}^3$ fixed, but changes the y, z-coordinates.



Enough to consider

$R_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$R_1(v_1) = v_1'$

Suppose $|v_1| = |v_1'| = d$

We know that

$$\begin{cases} y_0 = d\cos\varphi \\ z_0 = d\sin\varphi \end{cases} \quad \text{and} \quad \begin{cases} y_0' = d\cos(\pi/3 + \varphi) \\ z_0' = d\sin(\pi/3+\varphi) \end{cases}$$

$$\Longrightarrow \begin{cases} y_0' = \overbrace{d\cos\varphi}^{y_0}\cos\pi/3 - \overbrace{d\sin\varphi}^{z_0}\sin\pi/3 \\ z_0' = \underbrace{d\cos\varphi}_{y_0}\sin\pi/3 + \underbrace{d\sin\varphi}_{z_0}\cos\pi/3 \end{cases}$$

Recall $\cos\pi/3 = 1/2$, $\sin\pi/3 = \sqrt{3}/2$ $\Longrightarrow \begin{cases} y_0' = \frac{1}{2}y_0 - \frac{\sqrt{3}}{2}z_0 \\ z_0' = \frac{\sqrt{3}}{2}y_0 + \frac{1}{2}z_0 \end{cases}$

So $R_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is: $R_1(y,z) = \left(\frac{1}{2}y - \frac{\sqrt{3}}{2}z, \frac{\sqrt{3}}{2}y + \frac{1}{2}z\right)$

$$\Longrightarrow R(x,y,z) = \left(x, \tfrac{1}{2}y - \tfrac{\sqrt{3}}{2}z, \tfrac{\sqrt{3}}{2}y + \tfrac{1}{2}z\right)$$

- To find $[R]_{\mathcal{E}}$, we find:

$R(e_1) = R(1,0,0) = (1,0,0) = 1e_1 + 0e_2 + 0e_3 \Rightarrow [R(e_1)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$R(e_2) = R(0,1,0) = (0, \frac{1}{2}, \frac{\sqrt{3}}{2}) = 0e_1 + \frac{1}{2}e_2 + \frac{\sqrt{3}}{2}e_3 \Longrightarrow [R(e_2)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix}$

$R(e_3) = R(0,0,1) = (0, -\frac{\sqrt{3}}{2}, \frac{1}{2}) = 0e_1 - \frac{\sqrt{3}}{2}e_2 + \frac{1}{2}e_3$

$$\Longrightarrow [R(e_3)]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

4. cont'd: So $[R]_{\mathcal{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$

- $R(1,2,1) = \left(1,\ \tfrac{1}{2}(2) - \tfrac{\sqrt{3}}{2}(1),\ \tfrac{\sqrt{3}}{2}(2) - \tfrac{1}{2}(1)\right)$

$$= \left(1,\ 1 - \frac{\sqrt{3}}{2},\ \sqrt{3} - \frac{1}{2}\right).$$

5.

- $\mathcal{E} = \{ \overset{e_1}{\overset{\shortparallel}{(1,0)}}, \overset{e_2}{\overset{\shortparallel}{(0,1)}} \}$

- $T(1,0) = (1-0, 1+0) = \overset{1e_1 + 1e_2}{\overset{\shortparallel}{(1,1)}}$ $\quad [T(1,0)]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $T(0,1) = (0-1, 0+1) = \underset{-1e_1 + 1e_2}{\underset{\shortparallel}{(-1,1)}}$ $\quad [T(0,1)]_{\mathcal{E}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\Longrightarrow [T]_{\mathcal{E}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

- $\mathcal{B} = \{ \overset{u_1}{\overset{\shortparallel}{(1,1)}}, \overset{u_2}{\overset{\shortparallel}{(1,-1)}} \}$

- $T(1,1) = (1-1, 1+1) = (0,2) = a(1,1) + b(1,-1)$

$$\Rightarrow \begin{cases} a+b = 0 \\ a-b = 2 \end{cases} \Longrightarrow a=1 \; b=-1$$

So $T(1,1) = 1\,(1,1) + (-1)(1,-1) = 1u_1 + (-1)u_2$

$$\Longrightarrow [T(1,1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

- $T(1,-1) = (1+1, 1-1) = (2,0) = a(1,1) + b(1,-1)$

$$\Longrightarrow \begin{cases} a+b = 2 \\ a-b = 0 \end{cases} \Longrightarrow a=1 \; b=1$$

So $T(1,-1) = 1u_1 + 1u_2$

$$\Longrightarrow [T(1,-1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$