MATH 1303 MIDTERM

VERSION B

FEBRUARY 2005

UNIVERSITY OF OTTAWA

SOLUTIONS

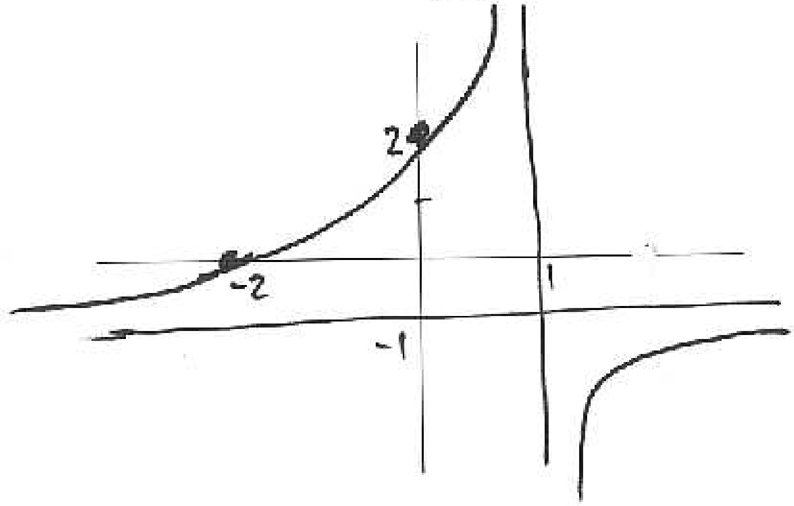
Name:

Problem	1	2	3	4	5	6	7	Total
Points	10	5	3	5	5	10	7	45
Score								

This exam contains 7 problems printed on 8 sheets of paper, including this front page.
The last page is scrap paper.
SHOW ALL YOUR WORK FOR FULL CREDIT!

1. **(10 points)** Analyze and graph the following function (find intercepts, local extreme points, inflection points, concavity, asymptotes, etc.).

$$f(x) = \frac{2+x}{1-x}$$



3'

$\lim_{x \to 1^+} f(x) = -\infty$

$x = 1$ ~~V~~ertical asymptote 1'

$f(-2) = 0$ 1'

$f(0) = 2$ 1'

$\lim_{x \to 1^-} f(x) = +\infty$

$y = 1$ ~~ver~~ horizontal asymptote 1'

$\lim_{x \to \infty} f(x) = -1$

$$f'(x) = \frac{(1-x) - (-1)(2+x)}{(1-x)^2} = \frac{1-x+2+x}{(1-x)^2} = \frac{3}{(1-x)^2} > 0$$ f increasing 1'

$$f''(x) = \frac{-3(-1)(1-x)(2)}{(1-x)^4} = \frac{6}{(1-x)^3}$$ 1'

	1	
f''	$+$	$-$
f'	$+$	$+$
f	↑ ∪	↑ ∩
	$+\infty$	$-\infty$

1'

2. **(5 points)** Determine whether the following improper integral converges or diverges.

$$\int_0^\infty x^2 e^{-x}\,dx$$

$$\int_0^\infty x^2 e^{-x}\,dx = \lim_{c\to\infty}\int_0^c x^2 e^{-x}\,dx$$

$$= \lim_{c\to\infty}\left((-c^2-2c-2)e^{-c}+2\right) = 2$$

$$\int_0^c x^2 e^{-x}\,dx = -x^2e^{-x}\Big|_0^c + 2\int_0^c xe^{-x}\,dx$$

$u = x^2 \quad dv = e^{-x}dx$

$du = 2x\,dx \quad v = -e^{-x}$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= -c^2e^{-c} + 2\left[-xe^{-x}\Big|_0^c + \int_0^c e^{-x}\,dx\right]$$

$$= -c^2e^{-c} + 2\left[-ce^{-c} + \underbrace{(-e^{-x})\Big|_0^c}_{1-e^{-c}}\right]$$

$$= -c^2e^{-c} - 2ce^{-c} + 2 - 2e^{-c}$$

$$= e^{-c}(-c^2-2c-2)+2$$

Answer: Convergent

3. **(3 points)** Let $f(x)$ be a function such that

$$f'(x) = \frac{1}{x+2} \text{ and } f(-1) = 2.$$

Find the equation for $f(x)$.

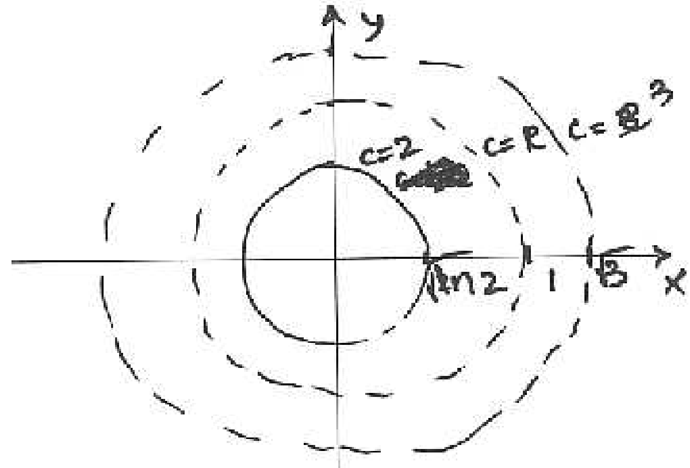
$f(x) = \int \frac{1}{x+2}\,dx = \ln|x+2| + C$

$f(-1) = 2 \Rightarrow \ln|-1+2| + C = 2 \Rightarrow C = 2$

Answer: $\boxed{f(x) = \ln|x+2| + 2}$

4. **(5 points)** Sketch the level curves for the function $f(x,y) = e^{x^2+y^2}$ for the values $c = 2, e, e^3$.

Compute &



$c=2 \quad e^{x^2+y^2} = 2 \Rightarrow x^2+y^2 = \ln 2$ → circle of radius $\sqrt{\ln 2}$ w/ center at origin

$c=e \quad e^{x^2+y^2} = e \rightarrow x^2+y^2 = 1$ → circle radius 1 center $(0,0)$

$c=e^3 \quad e^{x^2+y^2} = e^3 \rightarrow x^2+y^2 = 3$ 〃 〃 $\sqrt{3}$ 〃 〃

5. **(5 points)** Suppose

$$g(x,y) = \ln(\frac{1}{2x+3y}).$$

Find g_x and g_y. Is $(0,0)$ a critical point of g? Why?

$$g_x = \frac{-(2)\,/\,(2x+3y)^2}{1/(2x+3y)} = \frac{-2}{2x+3y}$$

$$g_y = \frac{-3\,/\,(2x+3y)^2}{1/(2x+3y)} = \frac{-3}{2x+3y}$$

(0,0) is not a critical pt, b/c although g_x (& g_y) are not defined at (0,0), (0,0) is not in the domain of g.

Answer:

6. **(10 points)** Find all the relative extreme points and saddle points for the function

$$f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - xy.$$

$f_x = x^2 - y$

$f_y = y^2 - x$

$f_x = 0 \Rightarrow x^2 = y$

$f_y = 0 \Rightarrow y^2 = x$

$\Rightarrow y^4 = y \Downarrow$ $y=0$ OR $y^3 = 1 \Rightarrow \boxed{y=1} \Rightarrow \boxed{x=1}$

$(0,0), (1,1)$ critical points

$f_{xx} = 2x$

$f_{yy} = 2y$

$f_{xy} = -1$

$*(1,1)$

$d = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (2)(2) - 1 = 3 > 0$

$\Rightarrow$ $f(1,1)$ is a relative (local) min.

$" \ 1/3 + 1/3 - 1 = -1/3$

because $f_{xx}(1,1) = 1 > 0$

and $d > 0$

$*(0,0)$ $d = f_{xx}(0,0)\, f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 0 - 1 = -1 < 0$

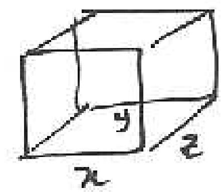
$(0,0,0)$ saddle pt.

Answer:

7. **(7 points)** Find the dimensions of a rectangular box of volume 1000 and minimum surface area.

Let the lengths of the sides be x, y, z

$$xyz = 1000 \Rightarrow z = \frac{1000}{xy}$$



$$S = 2xy + 2xz + 2yz$$

$$S = 2xy + (2x+2y)\left(\frac{1000}{xy}\right) = 2xy + \frac{2000}{y} + \frac{2000}{x}$$

$$S_x = 2y - \frac{2000}{x^2} = 0 \Rightarrow 2x^2y = 2000 \Rightarrow y = \frac{1000}{x^2}$$

$$S_y = 2x - \frac{2000}{y^2} = 0 \Rightarrow x = \frac{1000}{y^2} = \frac{1000}{\left(\frac{1000}{x^2}\right)^2} = \frac{x^4}{1000}$$

$$\Rightarrow x^3 = 1000 \Rightarrow \boxed{x = 10}$$

$$y = \frac{1000}{x^2} = \frac{1000}{100} = \boxed{10}$$

So $(10,10)$ is the only critical point, and must hence give us the minimum surface area.

We can check this:

$$S_{xx} = \frac{4000}{x^3}, \quad S_{yy} = \frac{4000}{y^3}, \quad S_{xy} = 2$$

$$d = \left(\frac{4000}{10^3}\right)\left(\frac{4000}{10^3}\right) - (2)^2 = 16 - 4 = 12 > 0$$

$S_{xx}(10,10) = 4 > 0$ So $(10,10)$ gives local min.

Answer: $x = 10, \; y = 10, \; z = 10$