Formulas/Constants, Physics/Oceanography 4505/5505 B Atmospheric Physics

Notes: There are some inconsistencies here with the text. There are also internal inconsistencies in the text. In Chapter 6, the density of pure water is referred to as ρ_l, but ρ_{water} is used in the index. I use ρ_w to refer to the density of water, while ρ_l and LWC refer to the liquid condensate density in a cloud. I use w_l to refer to liquid condensate mass mixing ratio, while the text uses w_l to refer to Liquid Water Content, which is really a density, and better designated ρ_l. It is bad practice for the text to use w to refer to water vapor mixing ratio, and w_l to refer to LWC, which are dimensionally different.

ρ_w: density of pure water
ρ_l: cloud liquid water density (usually called Liquid Water Content LWC)
$w_l = \rho_l/\rho_d$: cloud liquid water mass mixing ratio
$w_v = \rho_v/\rho_d$: vapor mass mixing ratio (often called w)

$N_A = 6.02 \times 10^{23}$ molecules/mole (Avogadro's number)
1 mb = 100 Pa
1 Pa = 1 N/m^2
$\Gamma_d = 9.8$ °C/km (dry adiabatic lapse rate)
$k = 1.38 \times 10^{-23}$ J/K
$\gamma = c_{pd}/c_{vd}$
$\gamma = 1.4$ (dry air)
$R_d = 287.05$ J/(kg K) (mean gas constant for dry air)
$M_d = 28.94$ g/mole (Mean molecular weight of dry air)
$p = e + p_d$ (total pressure equals dry air and vapor pressure)
$R_v = 461.5$ J/(kg K) (ideal gas constant for water vapor)
$\epsilon = M_v/M_d = R_d/R_v = 0.622$
$e = \rho_v R_v T$ (IGL for water vapor)
$p_d = \rho_d R_d T$ (IGL for dry air)
relative humidity: $RH = e/e_s \times 100\%$
specific humidity: $q = m_v/(m_d + m_v) = \rho_v/\rho = w/(1+w)$
water vapor mass mixing ratio: $w_v = \rho_v/\rho_d = \epsilon e/p_d = \epsilon e/(p-e)$
$e/p = w/(\epsilon + w)$
density of water $\rho_w = 1000$ kg/m^3
density of ice $\rho_i = 917$ kg/m^3

All temperatures T are in Kelvin. The temperature T_0 is the temperature at which the reference enthalpies h_{i0} and h_{l0} are defined. This is usually 0 °C or $T_0 = 273.15$ K.

Specific Heats

You can assume that all specific heats are independent of temperature.
$c_{vd} = 717.5$ J/(kg K) (specific heat at constant volume for dry air)
$c_{pd} = 1004.5$ J/(kg K) (specific heat at constant pressure for dry air)
ice: $c_i = 2106\ J/(kgK)$
water: $c_l = 4219\ J/(kgK)$
$c_{vv} = 1390$ J/(kg K) (at constant volume)

c$_{pv}$ = 1850 J/(kg K) (at constant pressure)

Specific Enthalpies

For an isobaric process with no external heating, the net heating Q equals the change in enthalpy.
dry air: $h_d = c_{pd}T$
$h_i = h_{i0} + c_i(T - T_0)$
$h_l = h_{l0} + c_l(T - T_0)$
$h_v = c_{pv}T$

Latent Heats

latent heat of fusion at 0 °C: $l_{f0} = 3.34 \times 10^5$ J/kg
T dependence of latent heat of fusion: $l_f(T) = h_l(T) - h_i(T)$
$l_f(T) = l_{f0} + (c_l - c_i)(T - T_0)$
$l_{f0} = h_{l0} - h_{i0}$
latent heat of vaporization at 0 °C: $l_{v0} = 2.5 \times 10^6$ J/kg
To get T dependence of latent heat of vaporization: $l_v(T) = h_v(T) - h_l(T)$
$l_v(T) = l_{v0} + (c_{pv} - c_l)(T - T_0)$
$l_{v0} = c_{pv}T_0 - h_{l0}$

Water vapor

M$_v$ = 18.015 g/mole (molecular weight of water vapor)
Mean molecular weight of moist air: $M = f_dM_d + f_vM_v$, where f_d and f_v are the molar fractions of dry air and the vapor ($f_d + f_v = 1$). (molar fractions equal pressure fractions, i.e. $f_v = e/p$).
Saturation vapor pressure (in hPa) over water (T in Kelvin): $e_s(T) = exp[21.653 - (5420/T)]$
dew point T_d from : $e = e_s(T_d)$
Saturation vapor pressure (in hPa) over ice (T in Kelvin, $e_{so} = 6.11$ hPa):

$$e_{si} = e_{so}exp[22.49 - (6142/T)]$$

specific humidity: $q = \rho_v/\rho = w_v/(1 + w_v) \approx \epsilon(e/p)$
water vapor mass mixing ratio: $w_v = \rho_v/\rho_d = \epsilon e/p_d = \epsilon(e/(p - e)) \approx \epsilon(e/p)$
total water mass mixing ratio: $w_t = w_v + w_l$ (parcel with liquid condensate and vapor)
$\epsilon = R_d/R_v = 0.622$
$e = w_vp_d/\epsilon$

Chapter 6: Cloud Microphysics

$$g_v = g_w + \frac{2\sigma}{r\rho_w}$$

Curvature effect: equilibrium vapor pressure $e_s(r, T)$ of a pure water droplet of radius r and temperature T:

$$e_s(r, T) = e_s(T)exp\left(\frac{2\sigma}{rR_v\rho_wT}\right)$$

where σ is the surface tension, ρ_w the density of water, R_v gas constant for water vapor, T temperature, and $e_s(T)$ the Clausius-Clapeyron saturation vapor pressure for water.

Approximate form of Kelvin equation:

$$\frac{e_s(r,T)}{e_s(T)} = exp\left(\frac{0.00108}{r}\right)$$

where r is the radius of a drop in microns.

Critical size r^* at which a pure water droplet becomes stable as a function of the ambient vapor pressure e and temperature T:

$$r^* = \frac{2\sigma}{R_v \rho_w T ln(e/e_s(T))}$$

Raoult's Law:

$$e_s(f_{H_2O}, T) = f_{H_2O}\, e_s(T) = e_s(T)\frac{N_{H_20}}{N_{H20} + N_X}$$

where N_X is the number of moles of X, and f_{H_2O} is the molar fraction of water in the liquid.

Raoult's Law for a curved droplet (i.e. Kohler curve):

$$\frac{e_s(f_{H_2O}, r, T)}{e_s(r,T)} = f_{H_2O}$$

where $e_s(r,T)$ is the saturation vapor pressure for a pure water droplet of radius r.

$$f_{H_2O} = \frac{\text{moles water}}{\text{moles water} + \text{moles dissolved ions}}$$

Kohler Curve:

$$\frac{e_s(f_{H_2O}, r, T)}{e_s(T)} = \left[exp\frac{2\sigma}{\rho_w R_v T r}\right]\left[1 + \frac{i m_x M_w}{M_x(\frac{4}{3}\pi r^3 \rho' - m_x)}\right]^{-1}$$

where M_w is the molecular weight of water (g/mole), σ the surface tension of water, m_x the mass of dissolved material (in kg), M_x the molecular weight of the solute(g/mole), i the number of ions the solute dissolves into, ρ' the density of the solution (kg/m^3). This can also be written:

$$\frac{e_s(f_{H_2O}, r, T)}{e_s(T)} = \left[exp\frac{2\sigma}{\rho_w R_v T r}\right]\left[1 + \frac{i m_x M_w}{m_w M_x}\right]^{-1}$$

where m_w is the mass of water in the droplet. For dilute aerosols, $m_w \approx \frac{4}{3}\pi r^3 \rho_w$, where ρ_w is the density of liquid water.

Approximate Form of Kohler Curve:

$$\frac{e_s(f_{H_2O}, r, T)}{e_s(T)} \approx 1 + \frac{a}{r} - \frac{b}{r^3}$$

where (and note units of m_x and M_x as defined above are different):

$$a = \frac{2\sigma}{\rho_w R_v T} \approx \frac{3.3 \times 10^{-7}}{T}[\text{in units m}]$$

$$b = \frac{3im_x M_w}{4\pi\rho_w M_x} \approx \frac{4.3 \times 10^{-6} im_x}{M_x} [\text{in units m}^3]$$

Need discussion of unit strangeness here: I think mx and Mx have different units.
The critical radius r_c can be approximated $r_c = \sqrt{\frac{3b}{a}}$
The critical supersaturation S_c can be approximated $S_c = \sqrt{\frac{4a^3}{27b}}$

Rate of change in droplet radius due to condensational growth:

$$\frac{dr}{dt} = \frac{1}{r} \frac{D\rho_v(\infty)}{\rho_w e(\infty)} [e(\infty) - e(r)]$$

D is the molecular diffusion constant, r is the radius of the drop, ρ_v is the density of the vapor away from the drop, and ρ_v is the density of the vapor at the droplets surface. $e(\infty)$ refers to the ambient water vapor pressure e in the cloud (i.e. as long as you are a few droplet radii away from a cloud droplet), and $e(r)$ refers to the water vapor pressure at the surface of the droplet. Cloud droplets are usually large enough so that one can ignore the curvature effect, so that $e(r) \approx e_s(T)$, i.e. just the Clausius Clapeyron saturated vapor pressure. In this case, one can write:

$$r\frac{dr}{dt} = G_l S$$

where the saturation ratio is defined:

$$S = [e - e_s(T)]/e_s(T)$$

$$S_i = [e - e_{si}(T)]/e_{si}(T)$$

$$G_l = \frac{D\rho_v(\infty)}{\rho_w}$$

$$r(t) = \sqrt{r_0^2 + 2G_l St}$$

Cloud Droplet Terminal Fall Speed for $r \leq 20\mu m$:

$$\nu = \frac{2}{9}\frac{g\rho_w r^2}{\eta}$$

where η is the viscosity of air, ρ_w is the density of water, and g is the gravitational acceleration.

Parameterized Expressions for Terminal Velocity

$\nu = k_1 r^2$ for $r \leq 20\,\mu m$ $k_1 = 1.2 \times 10^8\, m^{-1}s^{-1}$
$\nu = k_3 r$ for $20\,\mu m \leq r \leq 600\,\mu m$ $k_3 = 6000\, 1/s$
$\nu = k_2\sqrt{r}$ for $r \geq 600\,\mu m$ $k_2 = 220\,\sqrt{m}/s$

Collision Efficiency:

$$E(r_1, r_2) = \frac{y^2}{(r_1 + r_2)^2}$$

Growth by Continuous Collision/Coalescence

M refers to the mass of the collector droplet, r_1 to the radius of the collector droplet, ν_1 to the velocity of collector droplet, and ν_2 is the velocity of the cloud droplets (assumed to be a constant). LWC is the Liquid Water Content (better called a cloud liquid water density in kg/m^3), E_c is the collection efficiency, E is the collision efficiency, and ρ_w is the density of pure water. To solve problems, we almost always set $\nu_2 = 0$.

$$\frac{dM}{dt} = \pi r_1^2(\nu_1 - \nu_2) \cdot LWC \cdot E_c$$

$$\frac{dr_1}{dt} = \frac{(\nu_1 - \nu_2) \cdot LWC \cdot E_c}{4\rho_w}$$

Collector droplet being vertically advected in a cloud and growing by collision/coalescence

A cloud droplet is being carried upward in a cloud having updraft velocity w. r_0 is the initial cloud droplet radius, r_H the final radius, and H is the final height. The initial height is 0 (assumed cloud base). For the case $\nu_1 >> \nu_2$:

$$H = \frac{4\rho_w}{LWC}\Big[\int_{r_0}^{r_H} \frac{w}{\nu_1 E} dr_1 - \int_{r_0}^{r_H} \frac{dr_1}{E}\Big]$$

E is the collision efficiency, LWC is the liquid water content, ν_1 is the terminal velocity of the collector drop, r_1 is the radius of the collector drop, H is the height above cloud base, and w is the cloud updraft speed. Under the above assumptions, the final radius of the droplet as it exits the cloud (H = 0) will be given by:

$$R = r_0 + w\int_{r0}^{R} \frac{dr_1}{\nu_1}$$

Marshall-Palmer Raindrop Distribution as a Function of Raindrop Radius r

$N(r) = N_0 e^{-\Lambda r}$, where r is the raindrop radius, and Λ is a parameter which depends on the rain rate R.

Radar reflectivity Z

$$Z = \int_0^\infty N(D) D^6 dD$$

Condensational growth of an ice crystal:

$$dM/dt = DC/\epsilon_0\left[\rho_v(\infty) - \rho_{vi}\right]$$

C/ϵ_0 is a shape dependent parameter, $\rho_v(\infty)$ is the background water vapor density in the cloud (i.e. a few radii away from any ice crystals), and ρ_{vi} is the water vapor density adjacent to the ice crystal, and D is a diffusion constant. Alternatively, we can express this in a more useful way,

$$dM/dt = (C/\epsilon_0)G_iS_i$$

,

where

$$G_i = D\rho_v(\infty)$$

$$S_i = (e(\infty) - e_{si})/e_{si}$$

$\rho_v(\infty)$ is the water vapor density of the cloud, and would usually be calculated from the ideal gas law for water vapor on the assumption the cloud is saturated (has $e = e_s$). Note that you are sometimes asked to find dr/dt of a disk or cylinder. You then have to express the mass M of the crystal in terms of a radius r, and find a relationship between dM/dt and dr/dt. In this course, the only ice crystal shape we consider is a disk, for which $C/\epsilon_0 = 8r$.

Convection and Buoyancy

The basic definition of the *buoyancy* of an air parcel in a cloud, with respect to the background atmosphere, is in terms of density.

$$B = g\frac{(\rho_b - \rho_{cl})}{\rho_b}$$

ρ_b is the density of the background atmosphere outside the cloud, and ρ_{cl} is the density of the cloud. The total density of an air parcel is the sum of the dry air, vapor, and condensate components.

$$\rho_t = \rho_d + \rho_v + \rho_l$$

You can solve for the dry air density using the IGL for dry air, $p_d = \rho_d R_d T$, and similarly for water vapor $e = \rho_v R_v T$. For the condensate density ρ_l (which the book calls liquid water content LWC), you would have to be given some value, or told to ignore it. For the background atmosphere, can assume $\rho_l = 0$.

CAPE:

$$CAPE = \int_{LFC}^{LNB} B_p dz$$

This is the most basic form of CAPE. The integrated buoyancy work on an air parcel from the Level of Free Convection (LFC) to the Level of Neutral Buoyancy (LNB). Note that the parcel buoynacy $B_p > 0$, by definition of the LFC and the LNB.

$$CIN = -\int_{z=0}^{LFC} B_p dz$$

The buoyancy work you have to add to an air parcel before it becomes positively buoyant. $B_p < 0$ below the LFC.

Boundary Layer Physics

Richardson Number:

$$R_i = \frac{g}{T} \frac{\partial\overline{\theta}/\partial z}{\left(\partial\overline{u}/\partial z\right)^2 + \left(\partial\overline{v}/\partial z\right)^2}$$

Local heating due to turbulent heat flux divergence:

$$\frac{\partial\overline{\theta}}{\partial t} = -\frac{\partial\overline{w'\theta'}}{\partial z}$$

Kinematic Turbulent heat flux (K m/s): $F_H = \overline{w'\theta'}$
Dynamic Turbulent Heat flux (in W/m2): $Q_H = \rho c_p \overline{w'\theta'}$
Expression for net downward radiative flux: $F^* = F_{S\downarrow} - F_{S\uparrow} + F_{L\downarrow} - F_{L\uparrow}$
Net upward transfer of energy from earth's surface: $F_{net\uparrow} = -F^* + F_{Hs} + F_{Es}$
Expression for flux into ground: $F_{Gs} = -F_{net\uparrow}$
Bulk Aerodynamic Formulae:
Kinematic Heat Flux (in K m/s): $F_{Hs} = C_H |V| (T_s - T_{air})$
Dynamic Heat Flux (W/m^2): $F_{Hds} = \rho c_p F_{Hs}$
Kinematic Moisture Flux (in kg vapor/kg air × m/s): $F_{water} = C_E |V| (q_{sat}(T_s) - q_{air})$
Kinematic Latent Heat flux (K m/s): $F_{Es} = (L_v/c_p) F_{water}$
Dynamic Latent Heat flux (W/m^2): $F_{Eds} = \rho L_v F_{water}$
Bowen ratio: $B = F_{Hs}/F_{Es}$
downward momentum flux at the surface : $u_*^2 = C_D |V|^2 = -\overline{w'u'}$
Frictional acceleration due to turbulent momentum flux divergence:

$$\frac{D\overline{u}}{Dt} = -\frac{\partial\overline{w'u'}}{\partial z}$$

Dynamics

Cyclostrophic balance (assumption of direct equivalence of centripetal acceleration with pressure gradient acceleration):

$$\frac{v^2}{r} = (1/\rho)\left(\frac{\partial p}{\partial r}\right)$$

$$\text{Relative vorticity} : \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\text{Total vorticity} : \eta = \xi + f$$

Vorticity tendency from divergence (note a negative divergence is called a convergence):

$$\frac{d\eta}{dt} = -\eta \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

Approximately equivalent forms of divergence (note the difference between the pressure velocity ω and regular velocity w):

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\frac{d\omega}{dp} = -\frac{dw}{dz}$$

For solid body rotation at angular velocity ω: $\xi = 2\omega$
Gradient Wind Balance for steady circular motion about a regular low:

$$\frac{V^2}{r} = \frac{1}{\rho}\frac{dp}{dr} - fV$$