Formulas/Constants: PHYS/OCEA 4411/5411 A Atmospheric Dynamics I

$\Gamma_d = 9.8$ °C/km (dry adiabatic lapse rate)
$\gamma = c_p/c_v = 1.4$
R = 287.05 J/(kg K)
c_v = 717.5 J/(kg K)
c_p = 1004.5 J/(kg K)
g_0 = 9.81 m/s^2
$f = 2\Omega \sin\phi$
$\Omega = 7.292 \times 10^{-5}$ rad/sec
$\rho_0 = 1.225$ kg/m^3
$p_0 = 1013.25$ hPa
$M_d = 28.94$ g/mole (Mean molecular weight of dry air)
$\nu = 1.46 \times 10^{-5} m^2/sec$ (sea level)
$\Gamma_d = 9.8$ °C/km
$a = 6370$ km (radius of the earth)
specific volume: $\alpha = 1/\rho$
Rossby Number: $R_0 = U/fL$
Viscous Force per unit area A between two plates with relative velocity u_0 and separation l:

$$F = \frac{\mu A u_0}{l}$$

Viscous acceleration within a fluid in the x direction due to a shear in u in the z direction (ν is the kinematic viscosity):

$$a = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} = \nu \frac{\partial^2 u}{\partial z^2}$$

Angular momentum: $L = I\omega$, where the moment of inertia of a point mass m is $I = mR^2$ (R is the distance from the axis of rotation), $\omega = u/R$ is the angular velocity, and u is the tangential (zonal) velocity.

Momentum/unit mass about earths axis(with zonal speed u) : $M = L/m = (\Omega a \cos\phi + u)\cos\phi$

Relationship between local and total derivative:

$$\frac{\partial}{\partial t} = \frac{d}{dt} - \mathbf{U} \cdot \nabla$$

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\text{Relative vector vorticity}: \omega = \nabla \times \mathbf{U}$$

$$\text{Relative scalar vorticity}: \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\text{Total vs Local Derivative}: \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

$$\text{Pressure Gradient Acceleration}: \frac{\mathbf{F}}{m} = -\frac{1}{\rho}\nabla p = -\nabla_p \Phi$$

$$\text{Total Gravity}: \mathbf{g}(\phi) = \mathbf{g}^*(\phi) + \Omega^2 \mathbf{R}$$

$$\text{Definition of Geopotential}: d\phi = gdz = g_0 dZ$$

$$\phi(z) = \int_0^z gdz$$

$$Z = \frac{\phi(z)}{g_0}$$

$$\text{Hydrostatic Relationship}: \frac{dp}{dz} = -\rho g$$

$$\text{Momentum Equation (Z surface)}: \frac{d\mathbf{U}}{dt} = -2\mathbf{\Omega} \times \mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F_r}$$

Primitive equations:

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi - 2\Omega w \cos\phi + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + -2\Omega u \sin\phi + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u \cos\phi + F_{rz}$$

$$\text{Ideal Gas Law}: p = \rho RT$$

Potential Temperature: $\theta = T(p_0/p)^{0.286} = T(p_0/p)^{R/c_p}$, where $p_0 = 10^5$ Pa

$$\frac{T}{\theta}\frac{d\theta}{dz} = \frac{dT}{dz} + \frac{g}{c_p} = \frac{dT}{dz} + \Gamma_d$$

$$\text{Buoyancy acceleration}: B = \frac{dw}{dt} = g\frac{(T - T_0)}{T_0} = g\frac{\theta - \theta_0}{\theta_0} = -N^2\delta z \ (small\ \delta z)$$

$$\text{Buoyancy frequency}: N^2 = g\frac{d\ln\theta}{dz} = \frac{g}{\theta}\frac{d\theta}{dz}$$

N is an angular frequency so the period of the oscillation is $T = 2\pi/N$

$$\text{Thickness Relationship}: \Delta Z = \frac{R < T >}{g_0} ln(p_1/p_2)$$

$$\text{Geostrophic wind (Z surface)}: \mathbf{V_g} = \mathbf{k} \times \frac{1}{\rho f}\nabla p$$

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

Approximate Prognostic equations (coordinate form on a Z surface):

$$\frac{du}{dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f(v - v_g)$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f(u - u_g)$$

Approximate Prognostic equation (vector form on a Z surface):

$$\frac{d\mathbf{V}}{dt} = -f\mathbf{k} \times \mathbf{V_{ag}}$$

$$\text{Continuity equation (Z surface)}: \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0$$

$$\text{Continuity equation (Z surface)}: \frac{1}{\rho}\frac{d\rho}{dt} + \nabla \cdot U = 0$$

$$\text{Thermodynamic equation}: c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = \dot{Q}$$

$$\text{Thermodynamic equation}: c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = \dot{Q}$$

$$\text{Mechanical Energy Equation}: \dot{Q} = \frac{d}{dt}\left[\frac{(u^2 + v^2 + w^2)}{2} + \Phi + c_vT + p\alpha\right] - \alpha\frac{\partial p}{\partial t} - \mathbf{V}\cdot\mathbf{F}$$

$$\text{Bernouilli Equation}: \frac{(u^2 + v^2 + w^2)}{2} + \Phi + c_vT + p\alpha = \text{constant}$$

$$\frac{d\theta}{dt} = \frac{\theta}{T}\frac{\dot{Q}}{c_p}$$

$$\frac{T}{\theta}\frac{\partial\theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} = -\Gamma + \Gamma_d$$

$$\text{Momentum Equation p surface}: \frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V} = -\nabla_{\mathbf{p}}\phi$$

$$\text{Definition of pressure velocity}: \omega = \frac{dp}{dt}$$

$$\text{Vector Geostrophic wind on pressure surface}: \mathbf{V_g} = \mathbf{k}\times\frac{1}{f}\nabla_{\mathbf{p}}\phi$$

$$u_g = -\frac{1}{f}\frac{\partial\phi}{\partial y}$$

$$v_g = \frac{1}{f}\frac{\partial\phi}{\partial x}$$

$$\nabla_{\mathbf{p}}\cdot\mathbf{V_g} \approx 0$$

$$\text{Continuity equation in pressure coordinates}: \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial\omega}{\partial p} = 0$$

$$\text{Continuity equation in pressure coordinates}: \nabla\cdot\mathbf{V_h} = -\frac{\partial\omega}{\partial p}$$

$$\text{Thermodynamic equation in pressure coordinates}: \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - \omega\sigma_p = \frac{\dot{Q}}{c_p}$$

$$\text{Approximate expression on a p surface}: \omega \approx \frac{(-\nabla T\cdot\mathbf{V_h})}{-\sigma_p}$$

$$\text{Approximate expression on a PT surface}: \omega \approx \mathbf{V_h}\cdot\nabla p$$

$$\text{Static stability}: \sigma_p = -\frac{T}{\theta}\frac{\partial\theta}{\partial p} = -\frac{\partial T}{\partial p} + \frac{RT}{c_p p} = \frac{\alpha}{c_p} - \frac{\partial T}{\partial p} = \frac{\Gamma_d - \Gamma}{\rho g}$$

If pressure levels have fixed z, or $\partial p/\partial t = 0$: $\omega = -\rho g w$

$$\text{Surface Pressure Tendency}: \frac{\partial p_s}{\partial t} \approx -\int_0^{p_s}(\nabla\cdot\mathbf{V})dp$$

$$\text{Kinematic Estimate of }\omega: \omega(p) = \omega(p_s) - \int_{p_s}^{p}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dp$$

$$\text{Natural coordinates}: \frac{dV}{dt} = -\frac{\partial \phi}{\partial s}$$

$$\text{Natural coordinates}: \frac{V^2}{R} = -\frac{\partial \phi}{\partial n} - fV$$

$$\text{Gradient wind speed}: V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial \phi}{\partial n}\right)^{\frac{1}{2}}$$

$$\frac{\partial v_g}{\partial p} = -\frac{R}{fp}\left(\frac{\partial T}{\partial x}\right)_p$$

$$\frac{\partial u_g}{\partial p} = \frac{R}{fp}\left(\frac{\partial T}{\partial y}\right)_p$$

$$\text{Thermal Wind in Vector Form}: \frac{\partial \mathbf{V_g}}{\partial p} = -\frac{R}{fp}\mathbf{k} \times \nabla_{\mathbf{p}} T$$

$$\text{Thermal Wind}: u_T = -\frac{R}{f}\left(\frac{\partial < T >}{\partial y}\right)_p \ln(\frac{p_0}{p_1})$$

$$\text{Thermal Wind}: v_T = \frac{R}{f}\left(\frac{\partial < T >}{\partial x}\right)_p \ln(\frac{p_0}{p_1})$$

$$\text{Thermal Wind}: u_T = -\frac{1}{f}\frac{\partial(\phi_1 - \phi_0)}{\partial y}$$

$$\text{Thermal Wind}: v_T = \frac{1}{f}\frac{\partial(\phi_1 - \phi_0)}{\partial x}$$

$$\text{Definition of Circulation}: C = \oint \mathbf{U} \cdot \mathbf{dl}$$

$$\text{Kelvins Circulation Theorem}: \frac{dC}{dt} = -\oint \frac{dp}{\rho}$$

$$\text{Relationship Absolute and Relative Circulation}: C_a = C + C_e$$

$$\text{Definition of Earths Circulation}: C_e = 2\Omega A_e = 2\Omega A \sin\phi$$

$$\text{Vector vorticity}: \omega = \nabla \times \mathbf{U}$$

$$\text{Relative scalar vorticity}: \xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\text{Total (Absolute) vorticity} : \eta = \xi + f$$

$$\text{Relationship} : \text{ Vorticity and Circulation} : \xi = \lim(A \to 0) \frac{\oint \mathbf{V} \cdot \mathbf{dl}}{A}$$

$$\text{Vorticity in Natural Coordinates} : \xi = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$

$$\text{Ertels Potential Vorticity} : PV = (\xi_\theta + f) \left(-g \frac{\partial \theta}{\partial p} \right)$$

$$\text{Vorticity Tendency Height Surface} : \frac{d(\xi + f)}{dt} = \text{Divergence} + \text{Tilting} + \text{Solenoidal}$$

$$\text{Divergence Term} = -(\xi + f) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$\text{Tilting Term} = -\left(\frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right)$$

$$\text{Solenoidal Term} = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial p}{\partial x} \right)$$

Scaled Vorticity Equation Height Surface :

$$\frac{d_h \xi}{dt} = -v \frac{\partial f}{\partial y} - f \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$\frac{d_h \xi}{dt} = \frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y}$$

Vorticity Equation Pressure Surfaces :

$$\frac{\partial \xi}{\partial t} = -\mathbf{V} \cdot \nabla(\xi + f) - \omega \frac{\partial \xi}{\partial p} - (\xi + f) \nabla \cdot \mathbf{V} + \mathbf{k} \cdot \left(\frac{\partial \mathbf{v}}{\partial p} \times \nabla \omega \right)$$

Potential Vorticity Equation :

$$\frac{\tilde{d} PV}{dt} = \frac{\partial PV}{\partial t} + \mathbf{V} \cdot \nabla_\theta PV = \frac{PV}{\sigma} \frac{\partial}{\partial \theta} (\sigma \dot{\theta}) + \sigma^{-1} \mathbf{k} \cdot \nabla_\theta \times \left(\mathbf{F}_r - \dot{\theta} \frac{\partial \mathbf{V}}{\partial \theta} \right)$$

$$\text{Potential Vorticity Incompressible Fluid} : PV = \frac{\eta}{h}$$