# Math 4020 - Solutions of Assignment 1 - Winter 2011.

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1. Let $w$ be a nonzero complex number, and $n$ be a positive integer. Present a complete list of $n$'th roots of $w$. Prove the following three facts about your list:

   (i) Every element of your list is an $n$'th root of $w$.

   (ii) Each element of your list is distinct.

   (iii) Your list is complete, i.e. it contains all the $n$'th roots of $w$.

   **Solution:** Let $w$ be a nonzero complex number, and $\theta = \mathrm{Arg}(w)$. Then

$$w = |w|(\cos\theta + i\sin\theta).$$

   For each $k \in \{0, \ldots, n-1\}$, define

$$w_k = |w|^{\frac{1}{n}}(\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})).$$

   We will prove (i), (ii) and (ii) which prove that the above list is a complete list of $n$'th roots of $w$.

   **Solution of (i):** For each $k \in \{0, \ldots, n-1\}$, we have:

$$\begin{aligned} w_k^n &= \left[|w|^{\frac{1}{n}}(\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n}))\right]^n \\ &= |w|(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)) = w. \end{aligned}$$

   **Solution of (ii):** Let $0 \leq k \neq k' \leq n-1$. If $w_k = w_{k'}$, then

$$|w|^{\frac{1}{n}}(\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})) = |w|^{\frac{1}{n}}(\cos(\frac{\theta + 2k'\pi}{n}) + i\sin(\frac{\theta + 2k'\pi}{n})),$$

   which implies that $\cos(\frac{\theta+2k\pi}{n}) = \cos(\frac{\theta+2k'\pi}{n})$ and $\sin(\frac{\theta+2k\pi}{n}) = \sin(\frac{\theta+2k'\pi}{n})$. Thus,

$$\frac{\theta + 2k\pi}{n} = \frac{\theta + 2k'\pi}{n} + 2\pi j, \quad \text{for some } j \in \mathbb{Z}.$$

   Therefore, $k - k' = jn$. Since $k, k' \in \{0, \ldots, n-1\}$, we have $-(n-1) \leq jn \leq n-1$. So, $j$ has to be 0, and $k = k'$ which is a contradiction.

**Solution of (iii):** Let $w' = |w'|(\cos(\psi) + i\sin(\psi))$ be an $n$'th root of $w$, i.e. we have

$$|w|(\cos(\theta) + i\sin(\theta)) = (|w'|(\cos(\psi) + i\sin(\psi)))^n = |w'|^n(\cos(n\psi) + i\sin(n\psi)).$$

Therefore $|w'|^n = |w|$ and $\theta + 2k\pi = n\psi$ for some $k \in \mathbb{Z}$. Therefore, $|w'| = |w|^{\frac{1}{n}}$ and $\psi = \frac{\theta+2k\pi}{n}$.
**Case 1:** Let $k \geq 0$. Let $k = nq + r$ with $0 \leq r < n$. Then

$$w' = |w|^{\frac{1}{n}}(\cos(\frac{\theta + 2r\pi}{n}) + i\sin(\frac{\theta + 2r\pi}{n})) = w_r,$$

which belongs to the above list.

**Case 2:** Let $k < 0$. Let $q'$ and $0 \leq r' \leq n-1$ be such that $-k = nq' + r'$. Then

$$w' = |w|^{\frac{1}{n}}(\cos(\frac{\theta + 2(n-r')\pi}{n}) + i\sin(\frac{\theta + 2(n-r')\pi}{n})) = w_{n-r'},$$

which belongs to the above list.

2. Let $\rho > 1$, and $z_0, z_1 \in \mathbb{C}$ be fixed. Prove that the set of all the complex points $z$ that satisfy the equation

$$|z - z_0| = \rho|z - z_1|$$

forms a circle in the complex plane. Find the center and radius of that cycle.

**Solution:** Since $|z - z_0|^2 = \rho^2|z - z_1|^2$, we have

$$\begin{array}{rcl} (z-z_0)\overline{(z-z_0)} & = & \rho^2(z-z_1)\overline{(z-z_1)} \\ & \Rightarrow & \\ z\overline{z} - z\overline{z_0} - z_0\overline{z} + z_0\overline{z_0} & = & \rho^2(z\overline{z} - z\overline{z_1} - z_1\overline{z} + z_1\overline{z_1}) \\ & \Rightarrow & \\ |z|^2 - 2\mathrm{Re}(\overline{z}z_0) + |z_0|^2 & = & \rho^2(|z|^2 - 2\mathrm{Re}(\overline{z}z_1) + |z_1|^2). \end{array}$$

Thus,

$$(\rho^2 - 1)|z|^2 + 2\mathrm{Re}(\overline{z}z_0) - 2\rho^2\mathrm{Re}(\overline{z}z_1) + \rho^2|z_1|^2 - |z_0|^2 = 0.$$

Therefore,

$$\begin{aligned}
0 &= (\rho^2-1)|z|^2 + 2\mathrm{Re}(\overline{z}z_0) - 2\rho^2\mathrm{Re}(\overline{z}z_1) + \rho^2|z_1|^2 - |z_0|^2 \\
&= (\rho^2-1)|z|^2 + 2\mathrm{Re}(\overline{z}(z_0-\rho^2 z_1)) + \rho^2|z_1|^2 - |z_0|^2 \\
&= (\rho^2-1)\left[|z|^2 + 2\mathrm{Re}(\overline{z}(\frac{z_0-\rho^2 z_1}{\rho^2-1})) + \frac{\rho^2|z_1|^2-|z_0|^2}{\rho^2-1}\right] \\
&= (\rho^2-1)\left[|z+\frac{z_0-\rho^2 z_1}{\rho^2-1}|^2 + \frac{\rho^2|z_1|^2-|z_0|^2}{\rho^2-1} - |\frac{z_0-\rho^2 z_1}{\rho^2-1}|^2\right] \\
&= (\rho^2-1)\left[|z+\frac{z_0-\rho^2 z_1}{\rho^2-1}|^2 + \frac{(\rho^2|z_1|^2-|z_0|^2)(\rho^2-1)-|z_0-\rho^2 z_1|^2}{(\rho^2-1)^2}\right] \\
&= (\rho^2-1)\left[|z+\frac{z_0-\rho^2 z_1}{\rho^2-1}|^2 - \frac{\rho^2|z_0-z_1|^2}{(\rho^2-1)^2}\right].
\end{aligned}$$

Thus,

$$|z+\frac{z_0-\rho^2 z_1}{\rho^2-1}| = \frac{\rho|z_0-z_1|}{(\rho^2-1)},$$

which is the equation of a circle with center $-\frac{z_0-\rho^2 z_1}{\rho^2-1}$ and radius $\frac{\rho|z_0-z_1|}{(\rho^2-1)}$.

3. Let $D$ be a proper subset of $\mathbb{C}$. Prove that

   (i) The set of boundary points of $D$ is the same as the set of boundary points of $\mathbb{C}\setminus D$.

   (ii) $D$ is open if and only if $D$ has no boundary points.

   (iii) $D$ is closed if and only if it includes all its boundary points.

   **Solution of (i):** Let $z_0 \in \mathbb{C}$. Then $z_0$ is a boundary point of $D$ if and only if

   $$\forall r>0,\ b_r(z_0)\cap D \neq \emptyset \ \text{ and } \ b_r(z_0)\cap(\mathbb{C}\setminus D)\neq\emptyset,$$

   if and only if

   $$\forall r>0,\ b_r(z_0)\cap (\mathbb{C}\setminus D) \neq \emptyset \ \text{ and } \ b_r(z_0)\cap(\mathbb{C}\setminus(\mathbb{C}\setminus D))\neq\emptyset,$$

   which is equivalent to the fact that $z_0$ is a boundary point of $\mathbb{C}\setminus D$.

   **Solution of (ii):** ($\Rightarrow$) Let $D \subsetneq \mathbb{C}$ be a proper subset of the complex plane, and $z_0 \in D$ be arbitrary. Since $D$ is open, the point $z_0$ is an interior point of $D$, i.e. there exists $r>0$ such that $b_r(z_0)\subseteq D$. This implies that $b_r(z_0)\cap(D\setminus\mathbb{C})=\emptyset$. Thus $z_0$ is not a boundary point of $D$.

($\Leftarrow$) Let $z_0 \in D$ be an arbitrary element of $D$. Since $D$ has no boundary point, there exists $r > 0$ such that

$$b_r(z_0) \cap D = \emptyset \text{ or } b_r(z_0) \cap (\mathbb{C} \setminus D) = \emptyset.$$

Note that $z_0 \in B_r(z_0) \cap D \neq \emptyset$. Thus, $B_r(z_0) \cap (\mathbb{C} \setminus D) = \emptyset$, i.e. $b_r(z_0) \subseteq D$ which implies that $z_0$ is an interior point of $D$, and $D$ is open.

**Solution (iii):** By part (i) we have $\partial D = \partial \mathbb{C} \setminus D$. Then, $D$ is closed if and only if $\mathbb{C} \setminus D$ is open, if and only if $(\mathbb{C} \setminus D) \cap \partial(\mathbb{C} \setminus D) = \emptyset$. But,

$$(\mathbb{C} \setminus D) \cap \partial(\mathbb{C} \setminus D) = (\mathbb{C} \setminus D) \cap \partial D = \emptyset,$$

which means that $\partial D \subseteq D$.

4. Let $f$ be a complex function whose real and imaginary components are $u$ and $v$, i.e. $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Let $z_0 = x_0 + iy_0$ and $w_0 = s_0 + it_0$ be two complex numbers. Prove that if $\lim_{x \to x_0, y \to y_0} u(x, y) = s_0$ and $\lim_{x \to x_0, y \to y_0} v(x, y) = t_0$ then $\lim_{z \to z_0} f(z) = w_0$.

**Solution:** Let $\epsilon > 0$ be given. Since $\lim_{x \to x_0, y \to y_0} u(x, y) = s_0$, there exists $\delta_1 > 0$ such that

$$|x - x_0| < \delta_1 \text{ and } |y - y_0| < \delta_1 \Rightarrow |u(x, y) - s_0| < \frac{\epsilon}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that

$$|x - x_0| < \delta_2 \text{ and } |y - y_0| < \delta_2 \Rightarrow |v(x, y) - t_0| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $|z - z_0| < \delta$, then $|x - x_0| \leq |z - z_0| < \delta$ and $|y - y_0| \leq |z - z_0| < \delta$ for $z = x + iy$. Thus,

$$\begin{aligned} |f(z) - f(z_0)| &= |u(x, y) - u(x_0, y_0) + i(v(x, y) - v(x_0, y_0))| \\ &\leq |u(x, y) - u(x_0, y_0)| + |v(x, y) - v(x_0, y_0)| \\ &\leq \epsilon, \end{aligned}$$

which implies that $\lim_{z \to z_0} f(z) = w_0$.

5. Suppose that $f$ and $g$ are complex functions, and $M$ and $L$ are complex numbers such that $\lim_{z\to z_0} f(z) = M$ and $\lim_{z\to z_0} g(z) = L$. Prove that $\lim_{z\to z_0}(fg)(z) = ML$.

**Solution:** Let $f(x+iy) = u_f(x,y) + iv_f(x,y)$ be the decomposition of $f$ into its real and imaginary components. Let $g(x+iy) = u_g(x,y) + iv_g(x,y)$ be the decomposition of $g$ into its real and imaginary components. Let $z_0 = x_0 + iy_0$. Since $\lim_{z\to z_0} f(z) = M$ and $\lim_{z\to z_0} g(z) = L$, we have

$$\begin{aligned}
\lim_{x\to x_0, y\to y_0} u_f(x,y) &= \mathrm{Re}(M),\\
\lim_{x\to x_0, y\to y_0} v_f(x,y) &= \mathrm{Im}(M),\\
\lim_{x\to x_0, y\to y_0} u_g(x,y) &= \mathrm{Re}(L),\\
\lim_{x\to x_0, y\to y_0} v_g(x,y) &= \mathrm{Im}(L).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Re}(\lim_{z\to z_0} f(z)g(z)) &= \lim_{x\to x_0, y\to y_0} \mathrm{Re}(f(z)g(z))\\
&= \lim_{x\to x_0, y\to y_0} (u_f(x,y)u_g(x,y) - v_f(x,y)v_g(x,y))\\
&= \mathrm{Re}(M)\mathrm{Re}(L) - \mathrm{Im}(M)\mathrm{Im}(L).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Im}(\lim_{z\to z_0} f(z)g(z)) &= \lim_{x\to x_0, y\to y_0} \mathrm{Im}(f(z)g(z))\\
&= \lim_{x\to x_0, y\to y_0} (u_f(x,y)v_g(x,y) + v_f(x,y)u_g(x,y))\\
&= \mathrm{Re}(M)\mathrm{Im}(L) + \mathrm{Im}(M)\mathrm{Re}(L).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{z\to z_0} f(z)g(z) &= \mathrm{Re}(\lim_{z\to z_0} f(z)g(z)) + i\mathrm{Im}(\lim_{z\to z_0} f(z)g(z))\\
&= \mathrm{Re}(M)\mathrm{Re}(L) - \mathrm{Im}(M)\mathrm{Im}(L) + i(\mathrm{Re}(M)\mathrm{Im}(L) + \mathrm{Im}(M)\mathrm{Re}(L))\\
&= ML.
\end{aligned}$$