Math 4020 - Solutions of Assignment 2 - Winter 2012.

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1. Use the ϵ-δ definition of continuity to prove that the function $f(z) = \overline{z} + z^3$ is continuous on $\mathbb{C}$. Find the largest subset of $\mathbb{C}$ on which f is differentiable. Reference any theorem that you use.

   Solution: Let $z_0 \in \mathbb{C}$ be arbitrary. We will show that f is continuous at z_0. First note that for $z \in b_1(z_0)$,

   $$|z^2 + z_0^2 + zz_0| \leq |z|^2 + |z_0|^2 + |z||z_0| \leq (|z_0| + 1)^2 + |z_0|^2 + |z_0|(|z_0| + 1),$$

   where we applied the triangle inequality. Let $\delta = \min\{\frac{\epsilon}{2((|z_0|+1)^2+|z_0|^2+|z_0|(|z_0|+1))}, \frac{\epsilon}{2}, 1\}$. Then for every $z \in b_\delta(z_0)$ we have

   $$\begin{aligned} |f(z) - f(z_0)| &= |\overline{z} + z^3 - \overline{z_0} - z_0^3| \\ &\leq |\overline{z} - \overline{z_0}| + |z^3 - z_0^3| \\ &= |z - z_0| + |z - z_0||z^2 + z_0^2 + zz_0| \\ &\leq \delta + \delta[(|z_0| + 1)^2 + |z_0|^2 + |z_0|(|z_0| + 1)] \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

   Thus f is continuous.

   As for differentiability, note that the real and imaginary components of f are

   $$u(x, y) = x^3 - 3xy^2 + x \quad \text{and} \quad v(x, y) = 3x^2y - y^3 - y,$$

   and

   $$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= 3x^2 - 3y^2 + 1 \\ \frac{\partial u}{\partial y}(x, y) &= -6xy \\ \frac{\partial v}{\partial x}(x, y) &= 6xy \\ \frac{\partial v}{\partial y}(x, y) &= 3x^2 - 3y^2 - 1. \end{aligned}$$

   If Cauchy-Riemann equations are satisfied, we should have $3x^2 - 3y^2 + 1 = 3x^2 - 3y^2 - 1$, which never holds. So f is never differentiable.

2. Prove that the function $f(z) = z\mathrm{Re}(z)$ is differentiable only at the point $z = 0$, and find $f'(0)$.

Solution: The real and imaginary parts of f are

$$u(x, y) = x^2 \quad \text{and} \quad v(x, y) = xy.$$

If f is differentiable at (x, y), then by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x}(x, y) = 2x = x = \frac{\partial v}{\partial y}(x, y) \text{ and } \frac{\partial v}{\partial x}(x, y) = y = 0 = -\frac{\partial u}{\partial y}(x, y).$$

Thus the Cauchy-Riemann equations are only satisfied at 0. Hence f is not differentiable at any point of $\mathbb{C} \setminus \{0\}$. Now observe that u, v and all their first order partial derivatives with respect to x and y are continuous on $\mathbb{C}$. So by the theorem on the sufficient condition of differentiability, f is differentiable at 0, and

$$f'(0) = \frac{\partial u}{\partial x}(0, 0) + i\frac{\partial v}{\partial x}(0, 0) = 0.$$

3. Prove that the Cauchy-Riemann equations are satisfied for the function $f(x + iy) = \sqrt{xy}$ at the point $z_0 = 0$, but the derivative of f at $z_0 = 0$ does not exist. Explain why this does not contradict the theorem on the sufficient condition for differentiability.

Solution: We need to show that

$$\frac{\partial u}{\partial x}(0, 0) = \frac{\partial v}{\partial y}(0, 0) \quad \text{and} \quad \frac{\partial u}{\partial y}(0, 0) = -\frac{\partial v}{\partial x}(0, 0).$$

Note that $u(x, y) = \sqrt{xy}$ and $v(x, y) = 0$. Thus clearly $\frac{\partial v}{\partial x}(0, 0) = \frac{\partial v}{\partial y}(0, 0) = 0$. Moreover,

$$\frac{\partial u}{\partial x}(0, 0) = \lim_{h \to 0, h \in \mathbb{R}} \frac{\sqrt{(h)(0)} - \sqrt{0}}{h} = 0.$$

Similarly, we can show that $\frac{\partial u}{\partial y}(0, 0) = 0$. Thus the Cauchy-Riemann equations are satisfied at zero.

To check whether the derivation of f exists at 0, we compute

$$\lim_{h \to 0, h \in \mathbb{C}} \frac{\sqrt{h_x h_y}}{h}$$

where $h = h_x + ih_y$. Note that this limit does not exist, because it takes different values as h approaches zero on different curves (compare the limit along the curve $h_x = 0$ and $h_y \to 0$, and the curve $h_x = h_y \to 0$.) Thus f is not differentiable at 0.

This does not contradict the theorem on sufficient condition for differentiability, because the partial derivatives of u with respect to x and y are not continuous around 0.

4. Let $D, \Omega \subseteq \mathbb{C}$ be domains. Show (with an $\epsilon - \delta$ proof) that if f is continuous on D, and g is continuous on Ω, and $f(D) \subseteq \Omega$, then the composition function $g \circ f$ is continuous on D as well.

   Solution: Let $z_0 \in D$ and $\epsilon > 0$ be given. Since g is continuous at $f(z_0)$, there exists $r > 0$ such that $|w - f(z_0)| < r$ implies that $|g(w) - g(f(z_0))| < \epsilon$. Moreover, since f is continuous at z_0, there exists $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < r$. Putting these two together, we conclude that if $|z - z_0| < \delta$ then $|g(f(z)) - g(f(z_0))| < \epsilon$. Therefore $g \circ f$ is continuous.

5. Let $D, \Omega \subseteq \mathbb{C}$ be domains. Show that if f is analytic on D, and g is analytic on Ω, and $f(D) \subseteq \Omega$, then the composition function $g \circ f$ is analytic on D, and the chain rule holds:

$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \forall z \in D.$$

   Solution: Let $z_0 \in D$ be fixed, and define

$$\phi(w) = \begin{cases} \frac{g(w) - g(f(z_0))}{w - f(z_0)} & w \neq f(z_0) \\ g'(f(z_0)) & w = f(z_0) \end{cases}$$

   Clearly ϕ is continuous on Ω. Observe that, if $z \neq z_0$, we have

$$\frac{g(f(z)) - g(f(z_0))}{z - z_0} = \phi(f(z)) \frac{f(z) - f(z_0)}{z - z_0}.$$

   Indeed, in the case of $f(z) \neq f(z_0)$, the above equality holds by the definition of ϕ, and if $f(z) = f(z_0)$ then both sides of the equality are equal to zero. Since ϕ is continuous on Ω, and f is continuous on D, by Question 2, $\phi \circ f$ is continuous on D as well, and we have

$$\lim_{z \to z_0} \phi(f(z)) = \phi(f(z_0)) = g'(f(z_0)).$$

Thus, by taking the limit of both sides, we get

$$\begin{aligned}
\lim_{z \to z_0} \frac{g(f(z)) - g(f(z_0))}{z - z_0} &= \lim_{z \to z_0} \phi(f(z)) \frac{f(z) - f(z_0)}{z - z_0} \\
&= \lim_{z \to z_0} \phi(f(z)) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
&= g'(f(z_0)) f'(z_0),
\end{aligned}$$

which finishes the proof.