# Math 4020/5020 - Solutions of Assignment 3 - Winter 2012.

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1. Let $\gamma$, $\gamma_1$ and $\gamma_2$ be piecewise smooth curves. Let $f : \mathbb{C} \to \mathbb{C}$ be continuous on the curves $\gamma$, $\gamma_1$ and $\gamma_2$. Prove that

   (i) $\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz$.

   (ii) $\int_{\gamma_1+\gamma_2} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$.

   (iii) If $\gamma$ is a closed curve, and $\gamma_c$ is the curve obtained from $\gamma$ by shift of parameter, then $\int_\gamma f(x)dx = \int_{\gamma_c} f(x)dx$.

   **Solution of Part (i):** Let $\gamma : [a, b] \to \mathbb{C}$ be the curve as above. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a piecewise smooth partition for $\gamma$. Recall that $-\gamma$ is parametrized as

$$-\gamma : [a, b] \to \mathbb{C},\ \ -\gamma(t) = \gamma(a + b - t).$$

   Therefore the above partition results in the partition $s_0 = a < s_1 = a+b-t_{n-1} < \ldots < s_{n-1} = a + b - t_1 < s_n = b$ of $-\gamma$ into smooth pieces, and

$$\begin{aligned}
\int_{-\gamma} f(z)dz &= \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} f(-\gamma(s))(-\gamma)'(s)ds \\
&= \sum_{i=1}^{n} \int_{s_{i-1}}^{s_i} f(\gamma(a+b-s))(-\gamma')(a+b-s)ds \\
&= \sum_{i=1}^{n} \int_{a+b-s_{i-1}}^{a+b-s_i} f(\gamma(t))(-\gamma')(t)d(-t) \\
&= \sum_{i=1}^{n} \int_{t_{n-i+1}}^{t_{n-i}} f(\gamma(t))\gamma'(t)dt \\
&= -\sum_{i=1}^{n} \int_{t_{n-i}}^{t_{n-i+1}} f(\gamma(t))\gamma'(t)dt \\
&= -\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(\gamma(t))\gamma'(t)dt \\
&= -\int_\gamma f(z)dz,
\end{aligned}$$

where we used the change of variable $t = a + b - s$, and the fact that $t_{n-i} = a + b - s_i$.

**Solution of Part (ii):** Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ be the curves as above. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ and $c = s_0 < s_1 < \ldots < s_{m-1} < s_m = d$ be piecewise smooth partitions for $\gamma_1$ and $\gamma_2$ respectively. Recall that $\gamma_1 + \gamma_2$ is parametrized as

$$\gamma_1 + \gamma_2 : [a, b + d - c] \to \mathbb{C},\ \gamma_1 + \gamma_2(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t - b + c) & t \in [b, b + d - c] \end{cases}$$

The above two partitions result in the partition

$$r_0 = a < \ldots < r_{n-1} = t_{n-1} < r_n = t_n = s_0+b-c < r_{n+1} = s_1+b-c < \ldots < r_{n+m} = s_m+b-c$$

of $\gamma_1 + \gamma_2$ into smooth pieces, and

$$\begin{aligned}
\int_{\gamma_1+\gamma_2} f(z)dz &= \sum_{i=1}^{m+n} \int_{r_{i-1}}^{r_i} f((\gamma_1 + \gamma_2)(s))((\gamma_1 + \gamma_2))'(s)ds \\
&= \sum_{i=1}^{n} \int_{r_{i-1}}^{r_i} f((\gamma_1 + \gamma_2)(s))(\gamma_1 + \gamma_2)'(s)ds \\
&+ \sum_{i=n+1}^{n+m} \int_{r_{i-1}}^{r_i} f((\gamma_1 + \gamma_2)(s))(\gamma_1 + \gamma_2)'(s)ds \\
&= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(\gamma_1(s))\gamma_1'(s)ds \\
&+ \sum_{j=1}^{m} \int_{s_{j-1}+b-c}^{s_j+b-c} f(\gamma_2(s - b + c))\gamma_2'(s - b + c)ds \\
&= \int_{\gamma_1} f(z)dz + \sum_{j=1}^{m} \int_{s_{j-1}}^{s_j} f(\gamma_2(t))\gamma_2'(t)dt \\
&= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,
\end{aligned}$$

where we used the change of variable $t = s - b + c$.

**Solution of Part (iii):** Let $\gamma : [a, b] \to \mathbb{C}$ be the curve as above. Let $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$. We have $\gamma = \gamma_1 + \gamma_2$. From the definition of shift of parameter, it is easy to see that $\gamma_c = \gamma_2 + \gamma_1$. Thus,

$$\int_{\gamma_c} f(z)dz = \int_{\gamma_2+\gamma_2} f(z)dz = \int_{\gamma_2} f(z)dz + \int_{\gamma_1} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz = \int_{\gamma} f(z)dz.$$

2. Let $\gamma_1$ and $\gamma_2$ be piecewise smooth curves.

   (i) Prove that $L(\gamma_1) = L(-\gamma_1)$.

   (ii) Prove that $L(\gamma_1 + \gamma_2) = L(\gamma_1) + L(\gamma_2)$.

   **Solution of Part (i):** Let $\gamma : [a, b] \to \mathbb{C}$ be a curve as above. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be a piecewise smooth partition for $\gamma$. Recall that $-\gamma$ is parametrized as

$$-\gamma : [a, b] \to \mathbb{C},\ -\gamma(t) = \gamma(a + b - t),$$

   with the piecewise smooth partition $s_0 = a < s_1 = a + b - t_{n-1} < \ldots < s_{n-1} = a + b - t_1 < s_n = b$. Then

$$\begin{aligned} L(-\gamma) = \int_a^b |(-\gamma)'(t)|dt &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |-\gamma'(a+b-t)|dt \\ &= \sum_{i=1}^{n} \int_{a+b-t_{i-1}}^{a+b-t_i} |\gamma'(s)|(-ds) \\ &= -\sum_{i=1}^{n} \int_{s_{n-i+1}}^{s_{n-i}} |\gamma'(s)|ds \\ &= \sum_{i=1}^{n} \int_{s_{n-i}}^{s_{n-i+1}} |\gamma'(s)|ds \\ &= L(\gamma), \end{aligned}$$

   where we used the change of variable $s = a + b - t$, and the fact that $t_{n-i} = a + b - s_i$.

   **Solution of Part (ii):** Let $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ be the curves as above. Let $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ and $c = s_0 < s_1 < \ldots < s_{m-1} < s_m = d$ be piecewise smooth partitions for $\gamma_1$ and $\gamma_2$ respectively. Recall that $\gamma_1 + \gamma_2$ is parametrized as

$$\gamma_1 + \gamma_2 : [a, b + d - c] \to \mathbb{C},\ \gamma_1 + \gamma_2(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t - b + c) & t \in [b, b + d - c] \end{cases}$$

The above two partitions result in the partition

$$r_0 = a < \ldots < r_{n-1} = t_{n-1} < r_n = t_n = s_0+b-c < r_{n+1} = s_1+b-c < \ldots < r_{n+m} = s_m+b-c$$

of $\gamma_1 + \gamma_2$ into smooth pieces, and

$$\begin{aligned}
L(\gamma_1+\gamma_2) &= \sum_{i=1}^{m+n} \int_{r_{i-1}}^{r_i} |(\gamma_1+\gamma_2)'(s)|ds \\
&= \sum_{i=1}^{n} \int_{r_{i-1}}^{r_i} |(\gamma_1+\gamma_2)'(s)|ds \\
&+ \sum_{i=n+1}^{n+m} \int_{r_{i-1}}^{r_i} |(\gamma_1+\gamma_2)'(s)|ds \\
&= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\gamma_1'(s)|ds \\
&+ \sum_{j=1}^{m} \int_{s_{j-1}+b-c}^{s_j+b-c} |\gamma_2'(s-b+c)|ds \\
&= L(\gamma_1) + \sum_{j=1}^{m} \int_{s_{j-1}}^{s_j} |\gamma_2'(t)|dt \\
&= L(\gamma_1) + L(\gamma_2),
\end{aligned}$$

where we used the change of variable $t = s - b + c$.

3. Let $f$ be a complex function which is analytic on a domain $\mathcal{U}$. Suppose $f'(z)$ is continuous on $\mathcal{U}$. Prove that $\int_\gamma f(z)f'(z) = 0$, for every piecewise smooth closed curve $\gamma$ that lies entirely in $\mathcal{U}$.

   **Solution:** Define the new function $g$ on $\mathcal{U}$ to be $g(z) = f(z)f'(z)$. Then $g$ is continuous on $\mathcal{U}$, and the function $\frac{f^2}{2}$ is an antiderivative of $g$ on $\mathcal{U}$. Then, by a theorem in the notes, $\int_\gamma f(z)f'(z) = 0$, for every piecewise smooth closed curve $\gamma$ that lies entirely in $\mathcal{U}$.

4. Let $f$ be an analytic function on the open disc $b_1(0)$. Assume that $f'$ is continuous and bounded on $b_1(0)$. Prove that there exists $C > 0$ such that

$$|f(z_1) - f(z_2)| \le C|z_1 - z_2| \quad \forall z_1, z_2 \in b_1(0).$$

**Solution:** The function $f'$ is continuous on $b_1(0)$, and has an antiderivative on $b_1(0)$. Therefore for every two points $z_1$ and $z_2$ in $b_1(0)$, the integral $\int_\gamma f'(z)dz$ is independent of the path $\gamma$ that extends from $z_1$ to $z_2$ and lies entirely in $b_1(z_0)$. Let $C > 0$ be a constant for which $|f'(z)| \leq C$ for every $z \in b_1(0)$. Let $\gamma$ be the straight line segment from $z_1$ to $z_2$. Then,

$$|f(z_2) - f(z_1)| = |\int_\gamma f'(z)dz| \leq CL(\gamma) = C|z_2 - z_1|.$$

5. Compute $\int_\gamma \frac{1}{z}dz$, where $\gamma$ is the circle of radius $r$ centred at the origin and oriented counter clockwise.
   **Solution:** Parametrize $\gamma$ as $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = re^{2\pi it}$. Then,

$$\int_\gamma \frac{dz}{z} = \int_0^1 \frac{1}{re^{2\pi it}} r(2\pi i)e^{2\pi it}dt = 2\pi i.$$