Math 4020/5020 - Solution of Assignment 4 - Winter 2012.

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1. Compute the following integrals.

   (i) $\int_\gamma \frac{e^{z^2}-z^3}{z+i}dz$, where γ is a piecewise smooth simple closed curve in the upper half-plane oriented positively.

   (ii) $\int_\gamma \frac{z^2e^{z^3}}{z^2+1}dz$, where γ is a piecewise smooth simple closed curve oriented positively.

   (iii) $\int_\gamma \frac{\overline{z}}{z^2}dz$, where γ is the circle of radius 1 centered at the origin and oriented positively.

   (iv) $\int_\gamma \frac{|z|e^z}{2z-1}dz$, where γ is the circle of radius 1 centered at the origin and oriented clockwise.

Solution of (i): The function $f(z) = \frac{e^{z^2}-z^3}{z+i}$ is analytic on $\mathbb{C}\setminus\{-i\}$. Since γ and its interior both lie in D, by Cauchy-Goursat theorem,

$$\int_\gamma \frac{e^{z^2}-z^3}{z+i}dz = 0.$$

Solution of (ii): The function $f(z) = \frac{z^2e^{z^3}}{z^2+1}$ is analytic on $\mathbb{C}\setminus\{-i, i\}$. Let Ω denote the inside of γ. Note that i and $-i$ do not lie on γ if the integral is well-defined. We consider the following cases:

Case 1: Assume that $i, -i \notin \Omega$. Then by Cauchy-Goursat theorem, $\int_\gamma \frac{z^2e^{z^3}}{z^2+1}dz = 0$.

Case 2: Assume that $-i \notin \Omega$ and $i \in \Omega$. Then by Cauchy's Integral formula,

$$\int_\gamma \frac{z^2e^{z^3}}{z^2+1}dz = \int_\gamma \frac{\frac{z^2e^{z^3}}{z+i}}{z-i} = 2\pi i(\frac{i^2e^{i^3}}{2i}) = -\pi e^{-i}.$$

Case 3: Assume that $i \notin \Omega$ and $-i \in \Omega$. Then by Cauchy's Integral formula,

$$\int_\gamma \frac{z^2e^{z^3}}{z^2+1}dz = \int_\gamma \frac{\frac{z^2e^{z^3}}{z-i}}{z+i} = 2\pi i(\frac{(-i)^2e^{(-i)^3}}{-2i}) = \pi e^{i}.$$

Case 4: Assume that $i, -i \in \Omega$. There exist $r_1 > 0$ and $r_2 > 0$ such that $\overline{b_{r_1}(i)}$ and $\overline{b_{r_2}(-i)}$ both lie in Ω. Let C_1 and C_2 denote the circles of radius r_1 and r_2

centered at i and $-i$ respectively. Then, by a theorem in the notes, since f is analytic in between the curves,

$$\int_\gamma \frac{z^2e^{z^3}}{z^2+1}dz = \int_{C_1} \frac{z^2e^{z^3}}{z^2+1}dz + \int_{C_2} \frac{z^2e^{z^3}}{z^2+1}dz = -\pi e^{-i} + \pi e^{i},$$

using Cases 2 and 3.

Solution of (iii): We first parametrize γ as $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = e^{2\pi it}$. Then

$$\int_\gamma \frac{\overline{z}}{z^2}dz = \int_0^1 \frac{e^{-2\pi it}}{e^{4\pi it}}(2\pi i)e^{2\pi it}dt = 2\pi i[\frac{e^{-4\pi it}}{-4\pi i}]_0^1 = 0$$

Solution of (iv): First note that $|z| = 1$ for every z on γ. Hence by Cauchy's integral formula,

$$\int_\gamma \frac{|z|e^z}{2z-1}dz = \int_\gamma \frac{e^z}{2z-1}dz = -\int_{-\gamma} \frac{e^z}{2z-1}dz = -\frac{1}{2}\int_{-\gamma} \frac{e^z}{z-\frac{1}{2}}dz = (2\pi i)(-\frac{1}{2}e^{\frac{1}{2}}) = -\pi ie^{\frac{1}{2}},$$

where $-\gamma$ is oriented positively.

2. Let f be a function analytic on the open disc $b_1(0)$ (i.e. the open disc centered at the origin of radius 1). Prove that if $f(b_1(0)) \subseteq b_1(0)$ then $|f'(0)| \leq 1$.

Solution: Let $0 < r < 1$ be arbitrary. Let γ_r denote the circle of radius r centered at the origin and oriented positively. Then f is analytic on and inside γ_r, so by generalized Cauchy's integral formula, we have

$$f'(0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^2}dz.$$

By the assumption, we know that $|f(z)| < 1$ for every $z \in b_1(0)$. Therefore,

$$\begin{aligned} |f'(0)| &\leq \frac{1}{2\pi}\max_{z\in\gamma_r}\left|\frac{f(z)}{z^2}\right| L(\gamma_r) \\ &< \frac{1}{2\pi}\frac{1}{r^2}(2\pi r) = \frac{1}{r}. \end{aligned}$$

Now, letting r approach 1 (from the left), we conclude that $|f'(0)| \leq 1$.

3. Let f be an entire function. Suppose that there exists an integer $n > 0$ such that the nth derivative of f, $f^{(n)}$, is identically zero on $\mathbb{C}$. Show that f must be a polynomial.

Solution: We prove the above claim by induction:
Basis of induction: Assume $n = 1$, i.e. f is an entire function such that f' is identically zero on $\mathbb{C}$. Then by a theorem in the notes, f should be a constant function, i.e. a polynomial of degree 0.
Induction hypothesis: Assume that the claim holds for $n = k$, i.e. if f is an entire function such that $f^{(k)}$ is identically zero then f is a polynomial of degree $k - 1$.
Induction step: Let f be an entire function such that $f^{(k+1)}$ is identically 0. Let $g = f'$. Then by a theorem in the notes, g is entire as well. Moreover, $g^{(k)}$ is identically 0. Hence by the induction hypothesis g is a polynomial of degree $k - 1$, i.e.

$$g(z) = \lambda_{k-1}z^{k-1} + \lambda_{k-2}z^{k-2} + \ldots + \lambda_1 z + \lambda_0,$$

where $\lambda_0, \ldots, \lambda_{k-1} \in \mathbb{C}$. Define the new function $h : \mathbb{C} \to \mathbb{C}$ to be

$$h(z) = \frac{\lambda_{k-1}}{k}z^k + \frac{\lambda_{k-2}}{k-1}z^{k-1} + \ldots + \frac{\lambda_1}{2}z^2 + \lambda_0 z.$$

Then h is an entire function, since it is a polynomial. Moreover, $(h - f)' = h' - f' = h' - g$ is identically 0 on $\mathbb{C}$. Hence $h - f = \mu$ for a constant μ in $\mathbb{C}$ by the induction basis. Thus,

$$f(z) = \frac{\lambda_{k-1}}{k}z^k + \frac{\lambda_{k-2}}{k-1}z^{k-1} + \ldots + \frac{\lambda_1}{2}z^2 + \lambda_0 z + \mu.$$

4. Let f be an entire function. Suppose that there exists $n \in \mathbb{N}$ and $K > 0$ such that $|f(z)| < K|z|^n$ for every z in $\mathbb{C}$. Prove that f has to be a polynomial.

Solution: We will show that $f^{(n+1)}$ is identically zero. Let z_0 be an arbitrary element of $\mathbb{C}$. Let C_r denote the circle of radius r centered at z_0 and oriented positively. By generalized Cauchy's integral formula we have:

$$f^{(n+1)}(z_0) = \frac{(n+1)!}{2\pi i}\int_{C_r} \frac{f(z)}{(z - z_0)^{n+2}}dz.$$

Hence

$$
\begin{aligned}
|f^{(n+1)}(z_0)| &\leq \frac{(n+1)!}{2\pi} \max_{z \in C_r} \frac{|f(z)|}{|z-z_0|^{n+2}} L(C_r) \\
&\leq r(n+1)! \max_{z \in C_r} \left(\frac{|f(z)|}{|z|^n} \frac{|z|^n}{|z-z_0|^{n+2}} \right) \\
&\leq rK(n+1)! \max_{z \in C_r} \frac{(|z-z_0| + |z_0|)^n}{|z-z_0|^{n+2}} \\
&\leq rK(n+1)! \frac{(r+|z_0|)^n}{r^{n+2}}.
\end{aligned}
$$

Now letting r approach infinity, we get $|f^{(n+1)}(z_0)| = 0$. Since z_0 is arbitrary, we conclude that $f^{(n+1)}$ is identically zero on $\mathbb{C}$. Thus by Question 3, f is a polynomial of degree n.

5. Let f be an entire function (i.e. f is analytic on $\mathbb{C}$). Suppose that there exists a constant $M > 0$ such that $|f(z)| \leq M$ for every z in $\mathbb{C}$. Prove that f is a constant function.

Solution: This is Liouville's theorem. Let $z \in \mathbb{C}$ be fixed. Then, since f is analytic on and inside C_R for every $R > 0$,

$$
f'(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{(w-z)^2} dz,
$$

where C_R is the circle of radius R centred at z oriented positively. We have,

$$
|f'(z)| \leq \frac{1}{2\pi} L(C_R) \frac{1}{R^2} \max_{z \in C_R} |f(z)| \leq \frac{1}{R} M,
$$

which approaches to zero as R tends to infinity. Thus $|f'(z)| = 0$, i.e. $f'(z) = 0$ for every $z \in \mathbb{C}$. Now by a theorem from the notes, this implies that f is a constant function.