Math 4020 - Solution of Assignment 5 - Winter 2012.

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1. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series of radius of convergence $R > 0$. Prove that the power series converges uniformly on $b_r(z_0)$ for every $r < R$.
   Solution: Fix $r < R$. We need to show that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
   $$|S(z) - S_m(z)| < \epsilon, \ \forall m \geq N \ \text{ and } \ \forall z \in b_r(z_0),$$
   where $S_m(z) = \sum_{n=0}^{m} a_n(z-z_0)^n$ and $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$.
   Let $\epsilon > 0$ be given. Fix $z_1 \in b_R(z_0)$ such that $|z_1 - z_0| = r$. Then the above power series is absolutely convergent at z_1, i.e. $\sum_{n=0}^{\infty} |a_n(z_1-z_0)^n|$ is convergent. Thus there exists $N > 0$ such that $\sum_{n=m}^{\infty} |a_n(z_1-z_0)^n| < \epsilon$ for every $m \geq N$. Now, for every $z \in b_r(z_0)$ and every $m \geq N$, we have
   $$\begin{aligned} |f(z) - \sum_{n=0}^{m} a_n(z-z_0)^n| &= |\sum_{n=m+1}^{\infty} a_n(z-z_0)^n| \leq \sum_{n=m+1}^{\infty} |a_n(z-z_0)^n| \\ &\leq \sum_{n=m+1}^{\infty} |a_n(z_1-z_0)^n| < \epsilon. \end{aligned}$$

2. "Analytic continuation is unique". Let D_1 and D_2 be two domains such that $D_1 \cap D_2 \neq \emptyset$ is a domain as well. Suppose f is an analytic function on D_1. A function g is called an analytic continuation of f into D_2 if g is analytic on D_2 and $f(z) = g(z)$ for $z \in D_1 \cap D_2$. Prove that for D_1, D_2 and f as above, there is a unique analytic continuation.
   Solution: Suppose not, i.e. assume that there are two distinct analytic functions g_1 and g_2 on D_2 such that
   $$f(z) = g_1(z) = g_2(z) \ \forall z \in D_1 \cap D_2.$$
   Thus, $g_1 - g_2$ is a nonzero analytic function on D_2 such that $(g_1 - g_2)(z) = 0$ for every $z \in D_1 \cap D_2$. But this is a contradiction with the fact that zeros of non-constant analytic functions are isolated.

3. Find the Taylor series expansion of the following functions. In each case, find the domain in which the expansion converges to the function.
   - $\cos(z) := \frac{1}{2}(e^{iz} + e^{-iz})$ about $z = 0$.
   - $\sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz})$ about $z = 0$.

- $\frac{2z}{z^2+9}$ about $z = 0$.
- $\sin(z^2)$ about $z = 0$.
- $z\cos(z)$ about $z = \frac{\pi}{2}$.

Solution of Parts (i) and (ii): Note that $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is the Taylor series of the exponential function, valid on $\mathbb{C}$. Therefore,

$$\begin{aligned}
\cos(z) &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{i^n + (-i)^n}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sin(z) &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\
&= \frac{1}{2i}\left(\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right) \\
&= \frac{1}{2i}\sum_{n=0}^{\infty} \frac{i^n - (-i)^n}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}.
\end{aligned}$$

Both of the above Taylor series are valid throughout $\mathbb{C}$, because the Taylor series of the exponential function is valid for the whole complex plane.

Solution of part (iii) Recall that $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, valid for all $z \in b_1(0)$. Thus,

$$\begin{aligned}
\frac{2z}{z^2+9} &= \frac{2z}{9}\left(\frac{1}{1-(-\frac{z^2}{9})}\right) \\
&= \frac{2z}{9}\sum_{n=0}^{\infty}(-\frac{z^2}{9})^n \\
&= \sum_{n=0}^{\infty} \frac{2(-1)^n}{9^{n+1}} z^{2n+1},
\end{aligned}$$

valid for every z such that $|\frac{z^2}{9}| < 1$, i.e. valid for every element of $b_3(0)$.

Solution of part (iv) From Question 2, we have $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ for every $z \in \mathbb{C}$. Hence,

$$\sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2},$$

valid for all $z \in \mathbb{C}$.

Solution of part (v) Recall that $\cos(z) = \sin(\frac{\pi}{2} - z) = -\sin(z - \frac{\pi}{2})$. Recall that $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$ for every $z \in \mathbb{C}$. Hence,

$$\begin{aligned} z\cos(z) &= -(\frac{\pi}{2} + (z - \frac{\pi}{2}))\sin(z - \frac{\pi}{2}) \\ &= -(\frac{\pi}{2} + (z - \frac{\pi}{2})) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z - \frac{\pi}{2})^{2n+1} \\ &= -\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z - \frac{\pi}{2})^{2n+1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (z - \frac{\pi}{2})^{2n+2}, \end{aligned}$$

valid for every $z \in \mathbb{C}$.

4. Find the Laurent series expansion of the function $f(z) = \frac{1}{z^5(z+2)}$ about the origin in all the possible domains.

Solution: The function f is analytic on $\mathbb{C}\backslash\{0, -2\}$. We can define the following two annular domains about f in which f is analytic:

$$D_1 = \{z \in \mathbb{C} : 0 < |z| < 2\} \quad \text{and} \quad D_2 = \{z \in \mathbb{C} : 2 < |z|\}.$$

On D_1: For every $z \in D_1$, we have $|\frac{z}{2}| < 1$, and therefore $\frac{1}{1-(-\frac{z}{2})} = \sum_{n=0}^{\infty} (-\frac{z}{2})^n$. Hence the Laurent series of f on D_1 is given as

$$f(z) = \frac{1}{z^5(z+2)} = \frac{1}{2z^5} \sum_{n=0}^{\infty} (-\frac{z}{2})^n = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} z^{n-5}, \quad \text{valid for } z \in D_1.$$

On D_2: For every $z \in D_2$, we have $|\frac{2}{z}| < 1$, and therefore $\frac{1}{1-(-\frac{2}{z})} = \sum_{n=0}^{\infty} (-\frac{2}{z})^n$. Hence the Laurent series of f on D_2 is given as

$$f(z) = \frac{1}{z^5(z+2)} = \frac{1}{z^6} \sum_{n=0}^{\infty} (-\frac{2}{z})^n = \sum_{n=0}^{\infty} (-1)^n 2^n z^{-n-6}, \quad \text{valid for } z \in D_2.$$

5. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \ldots = f^{(k)}(z_0) = 0$, then the function g defined below is analytic at z_0.

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{k+1}} & \text{if } z \neq z_0 \\ \frac{f^{(k+1)}(z_0)}{(k+1)!} & \text{if } z = z_0 \end{cases}$$

Solution: Since f is analytic at z_0, there exists $r > 0$ such that f is analytic on $b_r(z_0)$. Thus f has a Taylor series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

valid on $b_r(z_0)$, where $a_n = \frac{f^{(n)}(z_0)}{n!}$. Since $f(z_0) = f'(z_0) = \ldots = f^{(k)}(z_0) = 0$, the smallest power of z in the Taylor series of f is $k + 1$, i.e.

$$f(z) = a_{k+1}(z - z_0)^{k+1} + a_{k+2}(z - z_0)^{k+2} + \ldots.$$

Note that the radius of convergence of the above power series is at least r (by Taylor's Theorem). Define the new function h on $b_r(z_0)$ to be

$$h : b_r(z_0) \to \mathbb{C},\ g(z) = a_{k+1} + a_{k+2}(z - z_0) + a_{k+3}(z - z_0)^2 + \ldots.$$

Clearly h is analytic on $b_r(z_0)$. Moreover, $f(z) = (z - z_0)^{k+1} h(z)$ for every $z \in b_r(z_0)$, which implies that $h(z) = \frac{f(z)}{(z-z_0)^{k+1}}$ for every $z \in b_r(z_0) \setminus \{z_0\}$. In addition, $h(z_0) = a_{k+1}$ by the definition of h. Thus $h(z) = g(z)$ on $b_r(z_0)$, which implies that g is analytic at z_0 (since h is analytic at z_0).