# Math 4020 - Solutions of Assignment 6 - Winter 2012.

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1. Let $f(x+iy) = u(x,y) + iv(x,y)$ be an entire function. Assume that $u$ has an upper bound in the $xy$-plane (i.e. there exists $M \in \mathbb{R}$ such that $u(x,y) \leq M$ for every $(x,y) \in \mathbb{R}^2$). Prove that $u$ must be constant throughout the plane.

   **Solution:** Define the new function $g : \mathbb{C} \to \mathbb{C},\ g(z) = \exp(f(z))$. Since $f$ and exp are entire functions, the function $g$ is entire as well. Moreover, for every $z = x + iy$ in $\mathbb{C}$,

$$\begin{aligned} |g(z)| &= |\exp(f(z))| = |\exp(u(x,y) + iv(x,y))| = |\exp(u(x,y))||\exp(iv(x,y))| \\ &= \exp(u(x,y)) \leq e^M, \end{aligned}$$

   since the exponential function on the real line is increasing. Now, by Liouville's theorem, we conclude that $g$ is constant throughout $\mathbb{C}$. Hence $|g| = \exp(u)$, and therefore $u$, must be constant throughout $\mathbb{R}^2$.

2. Let $f$ denote the function $\sum_{n=0}^{\infty} z^n$. Determine the domain of $f$. Find an analytic continuation of $f$ to the domain $\mathbb{C} \setminus \{1\}$.

   **Solution:** The radius of convergence of $\sum_{n=0}^{\infty} z^n$ is 1. Moreover, for any point $z$ such that $|z| = 1$, the above sum diverges, since $\lim_{n\to\infty} |z|^n = 1 \neq 0$. Thus $b_1(0)$ is the domain of definition of $f$. It is easy to see that the function $h(z) = \frac{1}{1-z}$ is the analytic continuation of $f$ to $\mathbb{C} \setminus \{1\}$.

3. (i) Find **all** the roots of the equation $\sin z = 0$ in the complex plane. Support your answer.

   (ii) Let $f(z) = \frac{1}{\sin(\frac{\pi}{z})}$. Find all the singularities of $f$, and determine whether each singularity is an isolated singularity or not.

   (iii) Let $g(z) = \frac{1}{\sin(\pi z)}$. Find all the singularities of $g$. For each singularity, determine if it is a pole, a removable, or an essential singularity. For each pole, find the order of the pole and the residue of $f$ at that pole.
   **Hint:** To determine the type of singularity at $z_0$, try to factor $\sin(\pi z)$ by $z - z_0$.

   (iv) Compute $\int_\gamma \frac{dz}{\sin(\pi z)}$, where $\gamma$ is a circle of radius 4.5 centered at the origin.

**Solution of (i):** Let $z = x + iy$ be a complex number. If $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 0$ then $e^{iz} = e^{-iz}$. In particular, we have $e^{-y} = |e^{iz}| = |e^{-iz}| = e^{y}$, which implies that $y = 0$ and $e^{ix} = e^{-ix}$. Hence

$$\cos(x) + i\sin(x) = e^{ix} = e^{-ix} = \cos(x) - i\sin(x),$$

which means that $\sin(x) = 0$. Hence $z = k\pi + i0 = k\pi$, $k \in \mathbb{Z}$, are all the zeros of $\sin(z)$.

**Solution of (ii):** Using part (i), it is clear that the singularities of $f$ are $\{\frac{1}{k} : k \in \mathbb{Z}\} \cup \{0\}$. The point 0 is the only singularity which is not isolated.

**Solution of (iii):** Using part (i), it is clear that the singularities of $g$ are all of the points of $\mathbb{Z}$. Let $k \in \mathbb{Z}$ be fixed, and consider the Taylor series of $\sin(\pi z)$ about $k$.

$$\sin(\pi z) = a_0 + a_1(z-k) + a_2(z-k)^2 + a_3(z-k)^3 + \ldots,$$

where $a_0 = \sin(k\pi) = 0$, $a_1 = \pi\cos(k\pi) = \pi(-1)^k$, .... Thus we get the following factorization:

$$\sin(\pi z) = (z-k)[a_1 + a_2(z-k) + a_3(z-k)^2 + \ldots] \quad \text{valid } \forall z \in \mathbb{C}.$$

Define the new function $h$ to be $h(z) = a_1 + a_2(z-k) + a_3(z-k)^2 + \ldots$. Then $h$ is entire, since the radius of convergence of the power series defining $h$ is $\infty$ (because it is the same as the radius of convergence of the power series representation of sin which is entire). Moreover, $h(k) = a_1 \neq 0$. Thus

$$g(z) = \frac{1}{\sin(\pi z)} = \frac{\frac{1}{h(z)}}{z-k},$$

and $\frac{1}{h}$ is analytic and nonzero at $k$. We now conclude that $k$ is a pole of order 1. Moreover,

$$\text{Res}(g,k) = \frac{1}{h(k)} = \frac{1}{a_1} = \frac{1}{(-1)^k\pi}.$$

**Solution of part (iv):** By the residue theorem,

$$\begin{aligned}\int_\gamma \frac{dz}{\sin(\pi z)} = 2\pi i[\text{Rez}(g,0) &+ \text{Rez}(g,\pi) + \text{Rez}(g,-\pi) + \text{Rez}(g,2\pi) + \text{Rez}(g,-2\pi) \\ &+ \text{Rez}(g,3\pi) + \text{Rez}(g,-3\pi) + \text{Rez}(g,4\pi) + \text{Rez}(g,-4\pi)] = 2i,\end{aligned}$$

if $\gamma$ is oriented positively, and $\int_\gamma \frac{dz}{\sin(\pi z)} = -2i$ if $\gamma$ is oriented negatively.

4. Use the residue theorem to evaluate the following integrals:

   (i) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx$.

   (ii) $\int_{0}^{\infty} \frac{x^2}{x^6+1} dx$.

**Solution of part 1:** Let $f(z) = \frac{ze^{iz}}{z^4+1}$. The function $f$ has four isolated singularities:

$$z_1 = e^{\frac{\pi i}{4}},\ z_2 = e^{\frac{3\pi i}{4}},\ z_3 = e^{\frac{5\pi i}{4}} = -z_1, z_4 = e^{\frac{7\pi i}{4}} = -z_2.$$

Let $C_R$ be the semicircular curve in the upper half-plane from $R$ to $-R$, and $L_R$ denote the line segment from $-R$ to $R$. Consider the curve $\gamma_R = C_R + L_R$. If $R > 1$ then $z_1$ and $z_2$ are the only singularities of $f$ that lie in $\gamma_R$. Hence by the residue theorem,

$$\begin{aligned}
\int_{\gamma_R} f(z)dz &= 2\pi i[\text{Res}(f, z_1) + \text{Res}(f, z_2)] \\
&= 2\pi i \left[\frac{z_1 e^{iz_1}}{(z_1^2 - z_2^2)(z_1 - z_3)} + \frac{z_2 e^{iz_2}}{(z_2^2 - z_1^2)(z_2 - z_4)}\right] \\
&= 2\pi i \left[\frac{z_1 e^{iz_1}}{(i+i)(2z_1)} + \frac{z_2 e^{iz_2}}{(-2i)(2z_2)}\right] \\
&= \frac{\pi}{2}[e^{iz_1} - e^{iz_2}] = i\pi e^{-\frac{\sqrt{2}}{2}} \sin(\frac{\sqrt{2}}{2}).
\end{aligned}$$

We now show that $\int_{C_R} f(z)dz$ approaches 0 as $R$ goes to infinity. Indeed,

$$\begin{aligned}
|\int_{C_R} f(z)dz| &= |\int_0^{\pi} \frac{Re^{it}e^{iRe^{it}}}{R^4 e^{4it} + 1} iRe^{it}dt| \\
&\leq \int_0^{\pi} R\frac{Re^{-R\sin t}}{R^4 - 1}dt \leq (\frac{R^2}{R^4 - 1})\frac{\pi}{R},
\end{aligned}$$

which converges to zero as $R$ tends to infinity. Note that we used Jordan's Lemma in the last inequality.

Finally, let $R$ approach infinity, and observe that

$$i\pi e^{-\frac{\sqrt{2}}{2}} \sin(\frac{\sqrt{2}}{2}) = \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+1}dx = \int_{-\infty}^{\infty} \frac{x\cos(x)}{x^4+1}dx + i\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^4+1}dx.$$

By equating the imaginary pasts of the above equality, we get

$$\int_{-\infty}^{\infty} \frac{x\sin(x)}{x^4+1}dx = \pi e^{-\frac{\sqrt{2}}{2}} \sin(\frac{\sqrt{2}}{2}).$$

**Solution of Part (ii):** Let $f(z) = \frac{z^2}{z^6+1}$. The function $f$ has six isolated singularities:

$$z_1 = e^{\frac{\pi i}{6}},\ z_2 = e^{\frac{\pi i}{2}},\ z_3 = e^{\frac{5\pi i}{6}},\ z_4 = e^{\frac{7\pi i}{6}},\ z_5 = e^{\frac{9\pi i}{6}},\ z_6 = e^{\frac{11\pi i}{6}}.$$

Let $C_R$ be the semicircular curve in the upper half-plane from $R$ to $-R$, and $L_R$ denote the line segment from $-R$ to $R$. Consider the curve $\gamma_R = C_R + L_R$. If $R > 1$ then $z_1$, $z_2$, and $z_3$ are the only singularities of $f$ that lie in $\gamma_R$. Hence by the residue theorem,

$$\begin{aligned}
\int_{\gamma_R} f(z)dz &= 2\pi i[\text{Res}(f, z_1) + \text{Res}(f, z_2) + \text{Res}(f, z_3)] \\
&= 2\pi i[\frac{z_1^2}{(z_1^2 - z_2^2)(z_1^2 - z_3^2)(2z_1)} \\
&+ \frac{z_2^2}{(z_2^2 - z_1^2)(z_2^2 - z_3^2)(2z_2)} + \frac{z_3^2}{(z_3^2 - z_2^2)(z_3^2 - z_1^2)(2z_3)}] \\
&= 2\pi i[\frac{-i}{6} + \frac{i}{6} + \frac{-i}{6}] = \frac{\pi}{3}.
\end{aligned}$$

We now show that $\int_{C_R} f(z)dz$ approaches 0 as $R$ goes to infinity. Indeed,

$$|\int_{C_R} f(z)dz| \leq \max_{z \in C_R} |f(z)| L(C_R) \leq \frac{R^2}{R^6 - 1}(\pi R),$$

which converges to zero as $R$ tends to infinity.

Finally, let $R$ approach infinity, and observe that

$$\frac{\pi}{3} = \int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx,$$

hence

$$\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}.$$

5. Let $f$ be an analytic function on the open disc $b_1(0)$. Assume that $|f(z)| \leq 1$ for all $z \in b_1(0)$, and $f(0) = 0$.
   (i) Prove that $|f'(0)| \leq 1$.
   (ii) Prove that $|f(z)| \leq |z|$ for all $z \in b_1(0)$.
   (iii) Prove that if $|f(w)| = |w|$ for some nonzero $w \in b_1(0)$, then there exists $c \in \mathbb{C}$ such that $f(z) = cz$ for all $z \in b_1(0)$.

**Solution of Part (i):** Let $0 < r < 1$ be arbitrary. Let $\gamma_r$ denote the circle of radius $r$ centered at the origin and oriented positively. Then $f$ is analytic on and inside $\gamma_r$, so by generalized Cauchy's integral formula, we have

$$f'(0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(z)}{z^2}dz.$$

By the assumption, we know that $|f(z)| \leq 1$ for every $z \in b_1(0)$. Therefore,

$$\begin{aligned} |f'(0)| &\leq \frac{1}{2\pi}\max_{z\in\gamma_r}\left|\frac{f(z)}{z^2}\right| L(\gamma_r) \\ &\leq \frac{1}{2\pi}\frac{1}{r^2}(2\pi r) = \frac{1}{r}. \end{aligned}$$

Now, letting $r$ approach 1 (from the left), we conclude that $|f'(0)| \leq 1$.

**Solution of Part (ii):** Let $g(z) = \frac{f(z)}{z}$ for every $z \in b_1(0) \setminus \{0\}$. Note that $z_0 = 0$ is a removable singularity of $g$, since

$$\lim_{z\to z_0} g(z) = \lim_{z\to z_0}\frac{f(z)}{z} = \lim_{z\to z_0}\frac{f(z)-f(0)}{z} = f'(0).$$

Let $h$ be the analytic continuation of $g$ to $b_1(0)$. We need to show that $|h(z)| \leq 1$ for every $z \in b_1(0) \setminus \{0\}$. Let $r < 1$. Then by the maximum modulus principle,

$$\max_{z\in b_r(0)\setminus\{0\}} |h(z)| \leq \max_{z\in b_r(0)} |h(z)| \leq \max_{|z|=r}|h(z)| \leq \max_{|z|=r}\frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

Thus if $r \to 1^-$, we get $\max_{z\in b_1(0)\setminus\{0\}} |h(z)| \leq 1$.

**Solution of Part (iii):** First note that $h(0) = \lim_{z\to 0} h(z) = \lim_{z\to 0} g(z) = f'(0)$. Hence, by Part (ii), we have $|h(z)| \leq 1$ for every $z \in b_1(0)$.

6. **(MATH5020):** Suppose that $f$ is entire and non-constant. Show that the closure of the range if $f$ is the whole complex plane, i.e.

$$\overline{\{f(z) : z \in \mathbb{C}\}} = \mathbb{C}.$$

**Solution:** Suppose not, i.e. suppose that there exists $w \in \mathbb{C}$ and $r > 0$ such that $b_r(w) \cap \{f(z) : z \in \mathbb{C}\} = \emptyset$. Define the complex function $g$ to be $g(z) = \frac{1}{f(z)-w}$. Clearly, $g$ is analytic on $\mathbb{C}$. Moreover, for every $z \in \mathbb{C}$,

$$|g(z)| = |\frac{1}{f(z)-w}| \leq \frac{1}{r},$$

i.e. $g$ is bounded on the complex plane. Hence, by Liouville's theorem, $g$ is a constant function, therefore $f$ is constant, which is a contradiction.