Math 4020 - Assignment 6 - Winter 2012.

Deadline: Wednesday, March 28th

Late assignments will be penalized 10%.

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1. Let $f(x+iy) = u(x,y) + iv(x,y)$ be an entire function. Assume that u has an upper bound in the xy-plane (i.e. there exists $M \in \mathbb{R}$ such that $u(x,y) \le M$ for every $(x,y) \in \mathbb{R}^2$). Prove that u must be constant throughout the plane.

2. Let f denote the function $\sum_{n=0}^{\infty} z^n$. Determine the domain of f. Find an analytic continuation of f to the domain $\mathbb{C} \setminus \{1\}$.

3. (i) Find **all** the roots of the equation $\sin z = 0$ in the complex plane. Support your answer.

   (ii) Let $f(z) = \frac{1}{\sin(\frac{\pi}{z})}$. Find all the singularities of f, and determine whether each singularity is an isolated singularity or not.

   (iii) Let $g(z) = \frac{1}{\sin(\pi z)}$. Find all the singularities of f. For each singularity, determine if it is a pole, a removable, or an essential singularity. For each pole, find the order of the pole and the residue of f at that pole.
   Hint: To determine the type of singularity at z_0, try to factor $\sin(\pi z)$ by $z - z_0$.

   (iv) Compute $\int_\gamma \frac{dz}{\sin(\pi z)}$, where γ is a circle of radius 4 centered at the origin.

4. Use the residue theorem to evaluate the following integrals:

   (i) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx$.

   (ii) $\int_0^{\infty} \frac{x^2}{x^6+1} dx$.

5. Let f be an analytic function on the open disc $b_1(0)$. Assume that $|f(z)| \le 1$ for all $z \in b_1(0)$, and $f(0) = 0$.

   (i) Prove that $|f'(0)| \le 1$.

   (ii) Prove that $|f(z)| \le |z|$ for all $z \in b_1(0)$.

   (iii) Prove that if $|f(w)| = |w|$ for some $w \in b_1(0)$, then there exists $c \in \mathbb{C}$ such that $f(z) = cz$ for all $z \in b_1(0)$.

6. **(MATH5020):** Suppose that f is entire and non-constant. Show that the closure of the range if f is the whole complex plane, i.e.

$$\overline{\{f(z) : z \in \mathbb{C}\}} = \mathbb{C}.$$

   Note: This question is considered as a bonus question for MATH4020.

Comments: The submitted solutions must be tidy and legible. You are to provide full solutions to the problems. You are allowed, and encouraged to collaborate with your classmates, but the write-ups should be done individually, without access to the papers of fellow students. Copying assignments or tests from any source, completely or partially, allowing others to copy your work, will not be tolerated.