Math 4020/5020 -Practice Questions - Winter 2012.

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1. Let f be an entire function. Prove that for every positive integer n, the following inequality holds (This is called Cauchy estimate).

$$|f^{(n)}(z_0)| \le \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|.$$

2. Show that there is a complex differentiable function defined on the set $\Omega = \{z \in \mathbb{C} : |z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

   Is there a complex differentiable function on Ω whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)}?$$

3. Let C be the circle of radius 1 centered at the origin. Suppose that f is analytic on and inside C, and satisfies $|f(z)| < 1$ for every $z \in C$. Show that the equation $f(z) = z^4$ has exactly four solutions (counting multiplicities) inside C.

4. Suppose that f is analytic on a domain containing $\{z : |z| \le 1\}$ and that $|f(e^{i\theta})| < 1$ for all $0 \le \theta \le 2\pi$. Show that there exists **exactly** one $z \in b_1(0)$ such that $f(z) = z$.

5. Let $g(z)$ be analytic in the open right half-plane $\{z : \mathrm{Re}(z) > 0\}$, and $|g(z)| < 1$. If $g(1) = 0$ how large can $g(2)$ be?

Comments: The submitted solutions must be tidy and legible. You are to provide full solutions to the problems. You are allowed, and encouraged to collaborate with your classmates, but the write-ups should be done individually, without access to the papers of fellow students. Copying assignments or tests from any source, completely or partially, allowing others to copy your work, will not be tolerated.