PART B.

(5) 1] Write down all the roots of the equation

$$1 + z + z^2 + \ldots + z^{n-1} = 0 .$$

(5) 2] Where is the function $|z+1|$ analytic? (Give reasons)

(10) 3] State and prove the fundamental theorem of algebra .

(10) 4] Write down, as explicitly as possible the Taylor expansion at zero of $(1+z)\arctan z$. What is its radius of convergence.

(10) 5] What is the radius of convergence of the Taylor expansion at zero of $\frac{z(1-z^2)}{\sin \pi z}$? Give reasons.

(10) 6] Suppose that

$$p_n(z) = a_0 + a_1 z + \ldots + a_n z^n$$

and suppose that

$$|a_i| > 2|a_{i-1}| \qquad i = 1,\ldots,n .$$

Show that all the zeros of p_n lie inside the unit disc $\{|z| < 1\}$.

(10) 7] Evaluate $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4 + a}$, $a > 0$

(10) 8] Suppose that f is entire and non-constant. Show that

$$\overline{\{f(z) | z \in \mathbb{C}\}} = \mathbb{C} .$$

(That is, show that the closure of the range is the whole complex plane).

(10) 9] Suppose that $\{p_n\}$ is a sequence of polynomials with positive coefficients. Suppose that $\{p_n\}$ converges to f pointwise on $\{|z| \le 1\}$. Show that f is analytic on $\{|z| < 1\}$ and show that the Taylor expansion of f has only non-negative coefficients .

(10) 10] Use the expansion

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2} .$$

(10) 11] Sketch a proof of the following statement. Every function which is meromorphic in the whole plane is the quotient of two entire functions.

6. Compute the value of the integral

$$\int_C \frac{dz}{z^2-1}$$

where C is the circle $|z| = 2$ described in the counterclockwise direction. What if C is the ellipse with foci 1 and -1?

7. ✓ Find the Laurent series expansion of the function $f(z) = \frac{1}{z^5(z+2)}$ in the region $0 < |z| < 2$.

8. Evaluate $\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx$ using residues of the function $\frac{z^2}{(z^2+1)^2}$.

9. Show that the set of rational numbers is (a) dense in $\mathbb{R}$; and (b) has Lebesgue measure zero.

10. Let f be the characteristic function of the set of rational numbers in [0,1]. Is f Riemann integrable? Lebesgue integrable? Why?

11. Find the sum of the series $\sum_{n=1}^{\infty} \frac{3n-2}{n^3+3n^2+2n}$.

12. Define the Cantor set. Show that it is closed, uncountable (don't need complete detail here), and of measure zero.

13. Define <u>pointwise</u> and <u>uniform</u> convergence for a sequence of functions $\{f_n\}$. If $\{f_n\}$ is a sequence of continuous functions which converges pointwise to 0 on the unit interval [0,1] and if, in addition, the convergence is is monotonic (i.e. $f_{n+1} \le f_n$ on $[0,1]\,\forall_n$) , then $\{f_n\}$ converges uniformly to 0 . (See next page.)

Complex Variables

1. Write $\dfrac{3+2i}{5-3i}$ in the form $a+ib$, $a,b \in \mathbb{R}$

2. Find all cube roots of 8.

3. For what values of $z \in \mathbb{C}$ is $\sum n z^n$ convergent?

4. If $f(z)$ is analytic and $\operatorname{Re} f(z) = x^3 - 3xy^2$, what is $\operatorname{Im} f(z)$?

5. Fundamental Theorem of Algebra

6. Find the MacLaurin series for $f(z) = \cos z$ and determine its radius of convergence.

7. Find the Laurent series around 0 for $f(z) = \dfrac{1}{z-z^2}$.

8. Cauchy's Integral Formula

9. Calculate $\displaystyle\int_0^\infty \frac{dx}{1+x^2}$ using residues.

Section B

6. Define the terms harmonic and holomorphic, and explain why the real part of a holomorphic function is harmonic.

7. Does there exist a holomorphic, non-constant function $f(z)$ with the properties

$$f(z+1) = f(z) \quad \text{and} \quad f(z+i) = f(z)$$

for all $z \in \mathbb{C}$? Why or why not?

8. a) Define the term meromorphic function and describe the Laurent series of such a function.

   b) Explain why $\frac{e^z}{z}$ is meromorphic but $e^{\frac{1}{z}}$ is not.

9. (a) Evaluate the integral

$$\int_{|z|=r} \frac{dz}{z(e^{2\pi i z} - 1)}.$$

   (b) Use (a) to show

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi}{6}.$$

10. (a) Define what is meant by a conformal mapping and state the Riemann mapping theorem concerning these

    (b) Explain why there is a conformal mapping of an equilateral triangle onto the upper half plane sending the vertices to $0, 1$ and ∞? Can you describe it explicitly?

Mathematics Department

Analysis Examination (Speciality)

Time: 3 hours

Answer as many questions as you are able. Full marks will be given for complete answers to 6 questions.

1) Determine the number of zeros of the polynomical

$$z^6 - 4z^5 + z^2 - 1$$

inside the unit circle.

2) Define the notion of an isolated singularity and discuss the behaviour of a function having such a singularity in a deleted neighbourhood of such a point.

3) State Liouville's Theorem. Using this theorem, prove the fundamental theorem of algebra.

4) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ on $[0,1]$. Show that the sequence $\{f_n\}$ converges pointwise to 0 but that it does not converge uniformly on $[0,1]$. State carefully a theorem which applies to this sequence and enables one to conclude that $\lim_{n\to\infty} \int_0^1 f_n(x) = 0$.

5) Discuss the concept of "completeness" in analysis as it arises in connection with metric spaces, ordered sets, and measures.

measurable functions on [0,1]). In a few words say why F can be extended to all of $\mathcal{L}^\infty[0,1]$ with the same norm.

(c) Consider the extended functional and still call it F. Show that there does not exist an L^1 function f with $F(g) = \int fg\,d\mu$ for all $g \in \mathcal{L}^\infty[0,1]$. (Hint: apply Lebesgue's Convergence Theorem and use $f_n(x) = x^n$.) (μ is the Lebesgue measure.)

(d) In four or five words, what does (c) say?

Part 2: Complex variables

Do as much of the following as you can in the remaining 90 minutes.

1. (a) Explain the terms differentiable, holomorphic, C^∞, meromorphic, analytic and entire. Discuss similarities and differences between the real and the complex case.

   (b) State and prove the Cauchy-Riemann equations.

2. (a) Find the singularities of

$$f(z) = \cos\left(\frac{1}{z}\right)\frac{e^z-1}{e^{2z}-1}.$$

   (b) What is the nature of these singularities?

   (c) For poles of f, find the orders and the residues.

3. (a) State Cauchy's Theorem.

   (b) Does Cauchy's Theorem have a converse?

4. (a) State Cauchy's integral formulas.

   (b) Establish the Cauchy estimates

$$|f^{(n)}(z_0)| \le \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)|, \quad n = 0,1,2,\ldots$$

   (c) Suppose f is an entire function satisfying $|f(z)| \le A\,|z|^n$ for some positive constants A and n and for all sufficiently large $|z|$. Show that f must be a polynomial.

   (d) Derive the Fundamental Theorem of Algebra.

5. State and discuss <u>one</u> of the following theorems:

   (a) The Weierstrass factorization theorem.

   (b) The Riemann mapping theorem.

Part B: Complex Analysis

1. (a) State a sufficiently strong version of Cauchy's Theorem for a triangle that will enable you to give an easy proof of the (first) Cauchy Integral Formula.

   (b) Carefully state the Cauchy Integral Formula and prove it.

   (c) Outline how one normally begins the proof of Cauchy's Theorem: i.e., what is the first step?

2. (a) Define the term harmonic function.

   (b) Precisely what is the relation between real harmonic functions and analytic functions?

   (c) Give an example of a real harmonic function on a domain Ω which is not the real part of an analytic function on Ω. Prove it.

3. (a) Find a 1:1 conformal mapping f from the unit disc to itself such that $f(0) = \frac{1}{2}$.

   (b) Carefully state Schwarz's Lemma.

   (c) Use Schwarz's Lemma to show that every 1:1 conformal map of the disc onto itself is given by a linear fractional transformation. (Hint: modify f so that it satisfies the hypotheses of Schwarz's lemma and then consider both f and f^{-1}.)

4. (a) If a is an isolated singularity of the (otherwise) analytic function f define the <u>residue of f at a</u>, $\text{Res}(f;a)$.

(b) Let $f(z) = \dfrac{\cos z}{(z-1)^2}$. Calculate $\text{Res}(f;1)$.

(c) Carefully state the Residue Theorem.

5. (a) If u is a real-valued harmonic function in a disc Δ of radius ρ centered at z, then prove the formula

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z+re^{it})\,dt \quad \text{for } 0<r<\rho .$$

(Hint: use the Cauchy Integral Formula).

(b) Does this property characterize harmonic functions? State a theorem. What tool is used in the proof of this theorem?

Attempt all questions. They are all of equal value.

1. (a) Write down the Cauchy-Riemann conditions which are necessary for a complex function to be differentiable at a point.

   (b) At what points is the function $f(z) = \bar{z}$ differentiable.

   (c) State Cauchy's integral formula.

   (d) Evaluate $\displaystyle\int_{|z|=1} \frac{\bar{z}}{z+\frac{1}{2}}\,dz$. (Hint: If $|z| = 1$ then $\bar{z} = ?$)

2. Theorem: If f(z) is meromorphic in a simply connected domain D which contains a Jordan curve C and if f has no zeros or poles on C then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P.$$

   (a) Explain all the terms used in this theorem and the meaning of the symbols N, P.

   (b) Indicate how the theory of residues can be used to prove this theorem.

   (c) Use Rouché's Theorem (and more elementary things) to show that the polynomial $2z^5 + 8z - 1$ has one real root in $[0, 1]$ and 4 roots in the annulus $1 < |z| < 2$.

3. State Liouville's Theorem. Use this theorem to show that the functions sin and cos are not bounded on the imaginary axis $(x = 0)$. Show how this theorem is used to prove the Fundamental Theorem of Algebra.

4. Write a brief account of conformal mappings. Include (at least) the definition, the main facts about bilinear (Möbius) transformations, a statement of the Riemann mapping theorem and some of the uses to which conformal mapping can be put.

5. (a) Given that the Fourier series of

$$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$

is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nx - \frac{(-1)^n}{n}\sin(x)\right)$$

find the Fourier series of $f(x) = |x|$ and use it to show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

Ph.D. Analysis Exam

L. Paynter.

Answer as many questions as you have time for.

1. Suppose z_1, z_2, z_3 are three complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 1$. Show that z_1, z_2, z_3 are vertices of an equilateral triangle inscribed in the unit circle.

2. Use DeMoivre's formula, i.e., the identity

$$e^{ix} = \cos x + i \sin x ,$$

and partial sums of geometric series to write $\sum_{k=0}^{n} \cos(k\theta)$ and $\sum_{k=0}^{n} \sin(k\theta)$ in closed form.

3. Find necessary and sufficient conditions (on a,b,c,d) that the mapping

$$f(z) = \frac{az+b}{cz+d}$$

map the unit circle $\{z : |z| = 1\}$ onto itself.

4. Is the function

$$f(z) = f(x+iy) = e^x \cos y + x + i(e^x \sin y + y)$$

analytic everywhere? Why or why not?

5. Find all roots of the equation $\sin z = 0$ in the complex plane.

All this paper looks reasonable to me. I prefer the bit more "general" style of the Real Analysis rather than the very specific complex questions.

Analysis Examination

Full marks may be obtained by complete answers to 13 of the following 15 questions. In general a few complete answers carry more marks than fragmentary answers to many questions.

1. Find all cube roots of 8 .

2. For what values of $z \in \mathbb{C}$ is $\sum_{n=1}^{\infty} \frac{z^n}{n}$ convergent? Absolutely convergent?

3. Find the image of the set $\{z : 0 \le \operatorname{Re} z \le 1\}$ under the map $f(z) = e^z$.

4. Find an analytic function $f(z)$ such that $\operatorname{Re} f(z) = 3xy^2 - x^3$.

5. Write down the power series of $\sqrt{1+z}$ about $z = 0$ and find its radius of convergence.

6. Give the Laurent expansion of $\frac{1}{(z-1)(z-2)}$ in the annulus $\{z : 1 < |z| < 2\}$.

7. State the maximal modulus theorem .

8. Calculate $\int_0^{\infty} \frac{dx}{4+x^2}$ using residues .

9. State and prove, in detail, the Fundamental Theorem of Algebra.

10. Show that every countable subset of $\mathbb{R}$ is of measure zero. Give an example, with relevant proofs, of an uncountable set which is of measure zero.

11. State Fatou's lemma for a sequence of non-negative measurable functions defined on a measurable subset of the real line.

    Use this lemma to prove the Monotone convergence Theorem.

    Give an example to show that strict inequality can hold in Fatou's Lemma.

12. Define the term <u>absolutely continuous</u> as applied to real valued functions of a real variable.

    Give an example of a function which is continuous on [0,1] but not absolutely continuous there.

    State a theorem relating absolute continuity to indefinite integration.

1. Is there an analytic function with real part e^{x+y} ?

2. What is the image of the first quadrant under $f(z) = z^2$?

3. Is there an analytic function in the open unit disk with $f(0)=1$ and $|f(z)| < 1$ for $0 < |z| < 1$?

4. How would you find the coefficient of z^{-3} in the Laurent expansion of $\frac{1}{(z-1)(z-2)}$ for $1 < |z| < 2$?

5. If we know the zeros of a polynomial $p(z)$, what do we know about those of $p'(z)$?

1'. Define Lebesgue-measurable functions on $\mathbb{R}$. If $\{x : f(x) < r\}$ is measurable for all rational r, is f measurable?

2'. Does measuarbility for $|f|$ imply that for f ?

3'. Is Lebesgue measure complete?

4'. If f_n is continuous on $[0,1]$ and $0 \le f_n(x) \le 1$ for all n and all $x \in [0,1]$, does $f_n \to 0$ imply $\int_0^1 f_n(x)\,dx \to 0$?

5' When is $\int_a^x f' = f(x) - f(a)$?

1''. What is a **bounded (= continuous)** operator on a Hilbert space? **What is a compact (= completely continuous)** operator? Are these distinct objects in finite or infinite dimensions?

2''. What is the spectrum of an operator? Is it nonempty? **What is it** for a compact operator? **Hermitian** operator?

3''. Spectral theorem for normal operators.

4''. Similar normal operators are unitarily equivalent.

7. a) State the Argument Principle.

b) State Rouche's Theorem and derive it from the Argument Principle.

c) Prove that $z^4 + 6z + 3$ has all its zeros in $\{|z| < 2\}$ and three of its zeros in $\{1 < |z| < 2\}$.

8. a) Write down the power series expansion for $\dfrac{z^3}{(1 - z^4)^2}$.

Where does it converge?

b) Give an example of a function that is analytic in $\mathbb{C} - \{(1, 0)\}$ whose Taylor series at (1, 0) converges only at that point.

c) Prove that

$$\prod_{n=0}^{\infty} (1 + z^{2^n}) = \frac{1}{1 - z} \quad \text{for} \quad |z| < 1.$$

5. Define the sequence of functions f_n on $[0,1]$ as follows:

$$f_n(x) = \begin{cases} n & \text{if } x \leq 1/n \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Is } \int_0^1 \lim_n f_n = \lim_n \int_0^1 f_n?$$

If not, show exactly how the conditions of the Lebesgue (Dominated) Convergence Theorem are violated.

6. Is every Lebesgue measureable function f on $[0,1]$ such that

$$\frac{df}{dx} = 0 \text{ a.e. on } [0,1]$$

necessarily constant?

If so, give a proof. If not, what additional hypothesis is needed to imply that the function is constant?

7. Show that if $f : [0,1] \mapsto \mathbb{R}$ is monotone and absolutely continuous, then f maps sets of measure zero to sets of measure zero. Is the conclusion still true if the absolute continuity hypothesis is dropped?

8. Let K be a given compact subset of $\mathbb{R}$. Show that there exists a Borel measure on $\mathbb{R}$ whose support is K. (Support is the complement of the union of all open sets of measure zero.)

Section B

9. Show that if z_0 is an nth root of unity ($z_0 \neq 1$) then

$$\sum_{k=0}^{n-1} z_0^k = 0.$$

Hence (or otherwise) show that

$$\sum_{k=1}^{1997} \cos\left(\frac{2\pi k}{1998}\right) = -1.$$

10. Evaluate the line integrals

$$\int_{C_1} \bar{z}dz \quad \text{and} \quad \int_{C_2} \bar{z}dz$$

where C_1 is the straight line segment from $z = 1$ to $z = i$ and C_2 is the quarter circle from $z = 1$ to $z = i$.

Explain why these integrals are not equal.

11. If $f(z) = u(x,y) + iv(x,y)$ is an entire function and if $u(x,y) = 2x^2 - 3x - 2y^2 + 5$ find (the most general form) of $f(z)$.

12. Use Cauchy's integral formula to evaluate the integral

$$\int_C \frac{e^z\,dz}{(z-2)^n}$$

where C is the circle $|z| = 3$ and n is a positive integer.

13. State three theorems (without proofs) which you consider to be of key importance in complex variable theory. Give some of the reasons why you think they are so important.

14. Use the residue theorem to evaluate the integral

$$\int_C \frac{z\,dz}{2z^4 + 5z^2 + 2}$$

where C is the unit circle $|z| = 1$.

Use this result to show that

$$\int_0^{2\pi} \frac{d\theta}{1 + 8\cos^2\theta} = \frac{2\pi}{3}.$$

(Hint: parametrize C and use the definition of cosine.)

15. State the Hahn-Banach Theorem (in any form that you wish).

Use the Hahn-Banach Theorem to show that if $x \neq 0$ is a point in a normed linear space then there exists a continuous linear functional f such that $f(x) = \|x\|$.

16. Prove (explaining the details) that if T is a bounded linear operator on a Banach space and if $\|T\| < 1$ then $(I - T)^{-1}$ exists.

Section B

6. Define what it means for a function to be *harmonic* and explain why the real part of a holomorphic function is harmonic. What can you say about the converse?

7. Show that if f is entire and for $|z| \geq 1$ it satisfies $|f(z)| \leq |z|^{\alpha}$ for a fixed α, then f is a polynomial of degree at most α.

8. Show that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n} = \pi \cot(\pi z).$$

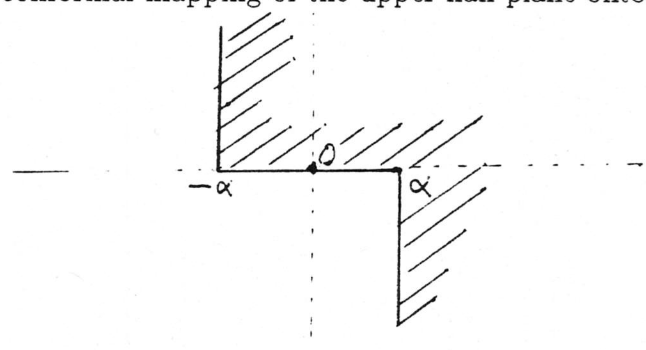
9. (a) Evaluate the integral

$$\int_{|z|=r} \frac{dz}{z(e^{2\pi i z}-1)}.$$

(b) Use (a) to show

$$\sum_{1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

10. Find a conformal mapping of the upper half-plane onto



which maps ∞ to ∞, 1 to α and -1 to $-\alpha$.