One-Way Analysis of Variance (ANOVA) is a method for comparing the means of a populations.

This kind of problem arises in two different settings

1. When a independent random samples are drawn from a populations.
2. When the effects of a different treatments on a homogeneous group of experimental units is studied, the group of experimental units is subdivided into a subgroups and one treatment is applied to each subgroup. The a subgroups are then viewed as independent random samples from a populations.

Notation:

	Population		Sample		
Group	Mean	SD	Size	Mean	SD
1	μ_1	σ	n_1	$\bar{x}_1$	s_1
2	μ_2	σ	n_2	$\bar{x}_2$	s_2
		$\ldots$			
a	μ_a	σ	n_a	$\bar{x}_a$	s_a

The hypotheses of interest in One-Way ANOVA are:

$$H_0: \quad \mu_1 = \mu_2 = ... = \mu_a$$
$$H_A: \quad \mu_i \neq \mu_j \text{ for some } i \neq j$$

Assumptions required for One-Way ANOVA

1. Random samples are independently selected from a (treatments) populations.
2. The a populations are approximately normally distributed.
3. All a population variances are equal.

The summary statistics and assumptions are the same assumptions as we made for the pooled t-test to compare two normal means, except that now we have $a \geq 2$ populataions.

Notation/terminology

a is the number of factor levels (treatments) or populations

x_{ij} is the jth observation in the ith sample, $j = 1, \ldots, n_i$

n_i is the sample size of the ith sample

$\bar{x}_{i.} = \sum_{j=1}^{n_i} x_{ij}/n_i$ is the ith sample mean

$s_i^2 = \frac{1}{(n_i-1)} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{i.})^2$ is the ith sample variance

$\bar{x}_{..} = \frac{1}{n} \sum_{i=1}^{a} n_i \bar{x}_{i.}$ is the overall mean of all observations

$n = \sum_{i=1}^{a} n_i$ is the total number of observations

Sums of squares and degrees of freedom

- The total variability in the response is called the total sum of squares, SST.
- The total variatiability SST is partitioned into between treatment and within treatment sums of squares.
- The notations SS_{Tr} (treatment sum of squares) and SSB (between sum of squares) are synonymous.
- The notations SS_{Error} and SSE (error sum of squares) and SSW (within sum of squares) are synonymous.
- Following are the forumlas for the **sums of squares**.

$$SST = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{..})^2$$

$$SS_{Tr} = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (\bar{x}_{i.} - \bar{x}_{..})^2 = \sum_{i=1}^{a} n_i(\bar{x}_{i.} - \bar{x}_{..})^2$$

$$SSE = \sum_{i=1}^{a} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_{i.})^2 = \sum_{i=1}^{a} (n_i - 1)s_i^2$$

- Associated with each sum of squares is its degrees of freedom.
  - The **total degrees of freedom** is $n-1$.
  - The **treatment degrees of freedom** is $a-1$.
  - The **error degrees of freedom** is $n-a$.
  - There is an additivity relationship in the sums of squares.

$$SST = SS_{Tr} + SSE$$

  - There is an additivity relationship for the degrees of freedom.

$$n-1 = (a-1) + (n-a)$$

  - Mean squares: $MSE = SSE/(n-a)$, $MS_{Tr} = SS_{Tr}/(a-1)$
  - Observed value of test statistic: $F_{obs} = MS_{Tr}/MSE$
  -

$$p - value = P(F_{a-1,n-a} \geq F_{obs})$$

Mean squares, F and p-values

Scaled versions of the treatment and error sums of squares (the sums of squares divided by their associated degrees of freedom) are known as **mean squares**: $MS_{Tr} = SS_{Tr}/(a-1)$ and $MSE = SSE/(n-a)$.

- MS_{Tr} and MSE are both estimates of the error variance, σ^2. MSE is always unbiased (its mean equals σ^2), while MS_{Tr} is unbiased only when the null hypothesis is true. When the alternative H_A is true, MS_{Tr} will tend to be larger than MSE.

- The ratio of the mean squares is $F = MS_{Tr}/MSE$. This should be close to 1 when H_0 is true, while large values of F provide evidence against H_0. The null hypothesis H_0 is rejected for large values of the observed test statistic F_{obs}.

- The **p-value** is the probability that an F random variable with $a-1$ numerator and $n-a$ denominator degrees of freedom is at least as large as F_{obs}, that is

$$p - value = P(F_{a-1,n-a} \geq F_{obs})$$

.

Calculations are conveniently displayed in an **ANOVA table**, as follows.

Source	df	SS	MS	F_{obs}	p-value
Treatments	$a-1$	SS_{Tr}	MS_{Tr}	$\frac{MS_{Tr}}{MSE}$	$P[F_{a-1,n-a} \geq F_{obs}]$
Error	$n-a$	SSE	MSE		
Total	$n-1$	SST			

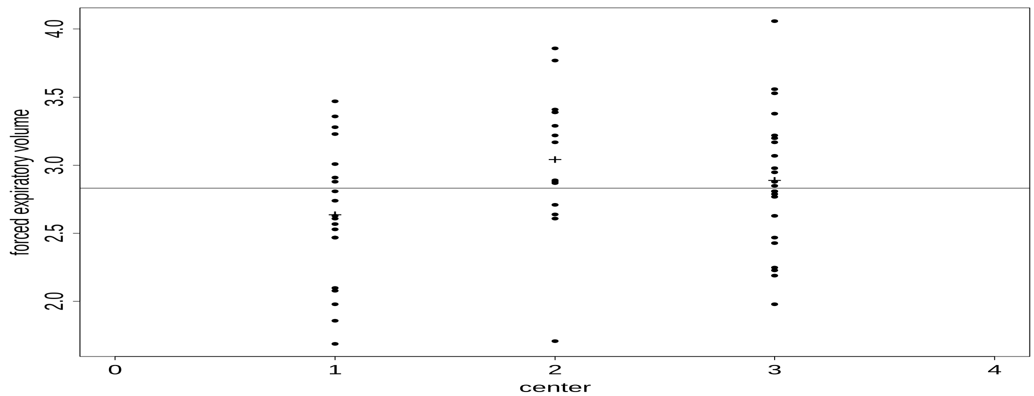
Review of F Distribution - using the F table

1. What is the probability that an F variable with 3 numerator and 5 denominator degrees of freedom is greater than 12.5? From the F table we see that $P(F_{3,5} > 12.06) = .01$ and $P(F_{3,5} > 33.20) = .001$, so that $.001 < P(F_{3,5} > 12.5) < .01$

Example: Pagano and Gauvreau gives the forced expiratory volume in 1 second for patients with coronary artery disease at three different centers.

	Johns Hopkins	Rancho Los Amigos	St. Louis	Overall
	3.23	3.22	2.79	
	3.47	2.88	3.22	
	1.86	1.71	2.25	
	2.47	2.89	2.98	
	3.01	3.77	2.47	
	1.69	3.29	2.77	
	2.10	3.39	2.95	
	2.81	3.86	3.56	
	3.28	2.64	2.88	
	3.36	2.71	2.63	
	2.61	2.71	3.38	
	2.91	3.41	3.07	
	1.98	2.87	2.81	
	2.57	2.61	3.17	
	2.08	3.39	2.23	
	2.47	3.17	2.19	
	2.47		4.06	
	2.74		1.98	
	2.88		2.81	
	2.63		2.85	
	2.53		2.43	
			3.20	
			3.53	
n_i	21	16	23	60
$\sum y_{ij}$	55.15	48.52	66.21	169.88
$\sum y_{ij}^2$	149.7581	151.2436	196.0483	497.05
$\bar{y}_i$	2.63	3.03	2.88	2.831
s_i	0.496	0.523	0.498	.522
SS_i	4.924	4.107	5.450	16.063

Does the mean FEV differ in the three groups?



- $H_0 : \mu_1 = \mu_2 = \mu_3$
  H_A : at least two of μ_1, μ_2, μ_3 are different
- Using calculations as above, can find $SSE = 14.479$, $SST = 16.063$.
- Then complete the ANOVA table, starting with $SS_{Tr} = 16.063 - 14.481 = 1.582$
- The completed ANOVA table is:

Source	Sum of Squares	DF	Mean Square	F
Between	1.582	2	0.791	3.11
Within	14.481	57	.254	
Total	16.063	59		

- degrees of freedom are 2 and 57. 57 is not in the table, so **go to next smaller df** = 40, and $p - value = P(F_{2,57} \geq 3.11) \approx P(F_{2,40} \geq 3.11) \in (.05, .1)$.
- Conclusion: reject H_0 at level α if and only $\alpha \geq .1$.

Following calculations were done using the R program, starting with summary statistics. Any differences from the numbers quoted above are due to rounding errors in the summary statistics.

```
> ni=c(21,16,23)             sample sizes
> ybari=c(2.63,3.03,2.88)   sample means
> si=c(.496,.523,.498)      sample standard deviations
> SSE=sum((ni-1)*si^2)      calculation of SSE
> SSE
[1] 14.47934
> ybar=sum(ybari*ni)/sum(ni)  calculation of overall mean
> ybar
[1] 2.8325            overall mean - note the rounding error
> SStr=sum(ni*(ybari-ybar)^2)  calculation of SSTr
> SStr
[1] 1.537125         SSTr has quite a bit of rounding error
```

A bit more detail on the underlying calculations

- The Total Sum of Squares is the sum of squared deviations from the overall average $SST = \Sigma\Sigma(y_{ij} - \bar{y})^2$ (generally done on computer)

$$(3.23 - 2.831)^2 + \ldots + (3.53 - 2.831)^2 = 16.063$$

- The Within Sum of Squares is the sum of squared deviations from the group averages $SSE = \Sigma\Sigma(y_{ij} - \bar{y}_i)^2$ (done on computer)

$$(3.23 - 2.63)^2 + \ldots + (3.22 - 3.03)^2 + \ldots + (3.53 - 2.88)^2$$

$$= 4.924 + 4.107 + 5.450 = 14.481$$

also calculated as a weighted sum of group variances $SSW = \Sigma(n_i - 1)s_i^2$ (easy calculation on hand calculator)

$$20(.496)^2 + 15(.523)^2 + 22(.498)^2 = 14.479$$

or as the sum of the within group sums of squares $SSW = \Sigma SS_i$

$$4.924 + 4.107 + 5.45 = 14.481.$$

- The Between Sum of Squares is the difference, $SSB = TSS - SSW$

$$16.063 - 14.481 = 1.582$$

also calculated as the weighted sum of squares of difference between group averages and the overall average $SSB = \Sigma n_i(\bar{y}_i - \bar{y})^2$ (easy calculation on hand calculator)

$$= 21(2.63 - 2.831)^2 + \ldots + 23(2.88 - 2.831)^2 = 1.537$$

(differences are due to round-off error).

- The degrees of freedom (DF) are the number of independent pieces of information.

- For Between, it is one less than the number of groups, $a - 1$.
- For Within, it is one less than the number in each group, summed over groups, which is the same as the sample size less the number of groups $n - a$.
- For Total, it is one less than the total number of observations, $n - 1$.

The Test

- If the means are not different, SSB will be a small component of the total, i.e. small relative to SSW.
- Under the stated assumptions

$$F = \frac{SSB/(a-1)}{SSW/(n-a)} = 3.11$$

  has an F distribution with $a - 1$ and $n - a$ degrees of freedom.
- Values of F near 1 or smaller indicate no difference.
- Tables give only selected quantiles of F for selected degrees of freedom.
- When the dfs we want aren't in the tables, we use the next smaller degrees of freedom.
- Because 3.11 is between 2.44 and 3.23, the .10 and .05 quantiles of the F with 2 and 40 degrees of freedom, we conclude P is between .05 and .10.
- The computer gives .052.

Which means are different. There are 3 possible comparisons. If we do 3 t-tests, each with probability of type I error α, then the probability of committing at least one type I error is greater than α. To control the overall probability of type I error at α, we can use the Bonferroni procedure, as follows.

- If we determine there are differences among the groups, we want to identify them, usually by testing each pair, $H_0 : \mu_i = \mu_l$ versus $H_a : \mu_i \neq \mu_l$.
- To stop the overall error rate from growing, we decrease the error rate on each test.
- The Bonferroni correction uses $\alpha^* = \alpha/c$ for each test, where $c = \binom{a}{2}$ is the number of possible comparisons.
- We use the pooled estimate of standard deviation, $s = \sqrt{MSE}$, and the test statistic
$$t_{i,k} = \frac{\bar{x}_i - \bar{x}_k}{s\sqrt{\frac{1}{n_i} + \frac{1}{n_k}}}$$
which has a t distribution with $n - a$ degrees of freedom.
- We find significant evidence against $H_0 : \mu_i = \mu_k$ at level α if $p - value \leq \alpha^*$.
- In the example, if we used $\alpha = .10$ we would conclude there were significant differences among the groups using the F test.
- $t_{1,2} = -2.39$, $t_{1,3} = -1.64$ and $t_{2,3} = .91$ have p-values .02, .11 and .37 (using the computer), only the first is less than $\alpha^* = .10/3 = .033$.