Chapter 2

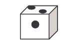
Probability

2.1 Sample Spaces and Events

Sample Space

The **sample space** of an experiment, denoted $\mathscr{S}$, is the set of all possible outcomes of that experiment.

Sample Space

Ex. Roll a die 

Outcomes: landing with a 1, 2, 3, 4, 5, or 6 face up.

Sample Space: $\mathscr{S}=\{1, 2, 3, 4, 5, 6\}$

Events

An **event** is any collection (subset) of outcomes contained in the sample space $\mathscr{S}$. An event is **simple** if it consists of exactly one outcome and **compound** if it consists of more than one outcome.

Relations from Set Theory

1. The **union** of two events A and B is the event consisting of all outcomes that are either in A or in B.

Notation: $A \cup B$

Read: A or B

Relations from Set Theory

2. The **intersection** of two events A and B is the event consisting of all outcomes that are in both A and B.

Notation: $A \cap B$

Read: A and B

Relations from Set Theory

3. The **complement** of an event A is the set of all outcomes in $\mathscr{S}$ that are not contained in A.

Notation: A'

Events

Ex. Rolling a die. $S = \{1, 2, 3, 4, 5, 6\}$

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

$A \cup B = \{1,2,3,5\}$

$A \cap B = \{1,3\}$

$A' = \{4,5,6\}$

Mutually Exclusive

When A and B have no outcomes in common, they are **mutually exclusive** or **disjoint** events

Mutually Exclusive

Ex. When rolling a die, if event $A = \{2, 4, 6\}$ (evens) and event $B = \{1, 3, 5\}$ (odds), then A and B are mutually exclusive.

Ex. When drawing a single card from a standard deck of cards, if event A = {heart, diamond} (red) and event B = {spade, club} (black), then A and B are mutually exclusive.

Venn Diagrams

2.2 Axioms, Interpretations, and Properties of Probability

Axioms of Probability

Axiom 1 $P(A) \geq 0$ for any event A

Axiom 2 $P(\mathscr{S}) = 1$

If all A_i's are mutually exclusive, then

Axiom 3 $P(A_1 \cup A_2 \cup \ldots \cup A_k) = \sum_{i=1}^{k} P(A_i)$ (finite set)

$P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$ (infinite set)

Properties of Probability

For any event A, $P(A) = 1 - P(A')$.

If A and B are mutually exclusive, then $P(A \cap B) = 0$.

For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Ex. A card is drawn from a well-shuffled deck of 52 playing cards. What is the probability that it is a queen or a heart?

Q = Queen and H = Heart

$P(Q) = \frac{4}{52},\ P(H) = \frac{13}{52},\ P(Q \cap H) = \frac{1}{52}$

$P(Q \cup H) = P(Q) + P(H) - P(Q \cap H)$

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

2.3

Counting Techniques

Product Rule

If the first element or object of an ordered pair can be used in n_1 ways, and for each of these n_1 ways the second can be selected n_2 ways, then the number of pairs is $n_1 n_2$.

** Note that this generalizes to k elements (k – tuples)

Permutations

Any ordered sequence of k objects taken from a set of n distinct objects is called a **permutation** of size k of the objects.

Notation: $P_{k,n}$

$$\boxed{P_{k,n} = n(n-1)\cdot\ldots\cdot(n-k+1)}$$

Factorial

For any positive integer m, $m!$ is read "m factorial" and is defined by $m! = m(m-1)\cdot\ldots\cdot(2)(1)$.

Also, $0! = 1$.

Note, now we can write:

$$\boxed{P_{k,n} = \frac{n!}{(n-k)!}}$$

Ex. A boy has 4 beads – red, white, blue, and yellow. How different ways can three of the beads be strung together in a row?

This is a permutation since the beads will be in a row (order).

$$P_{3,4} = \frac{4!}{(4-3)!} = 4! = 24$$

number selected

total

24 different ways

Combinations

Given a set of n distinct objects, any unordered subset of size k of the objects is called a **combination**.

Notation: $\binom{n}{k}$ or $C_{k,n}$

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

Ex. A boy has 4 beads – red, white, blue, and yellow. How different ways can three of the beads be chosen to trade away?

This is a combination since they are chosen without regard to order.

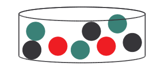
$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!} = 4$$

total

number selected

4 different ways

Ex. Three balls are selected at random without replacement from the jar below. Find the probability that one ball is red and two are black.



$$= \frac{\binom{2}{1}\cdot\binom{3}{2}}{\binom{8}{3}} = \frac{2\cdot 3}{56} = \frac{3}{28}$$

Examples for Section 2.3 : Counting Techniques

- Example1: A house owner doing some remodelling requires the services of both a plumbing contractor and an electrical contractor; there are 12 plumbing contractors and 9 electrical contractors, in how many ways can the contractors be chosen?
- Example 2: A family requires the services of both an obstetrician and a pediatrician. There are two accessible clinics, each having two obstetricians and three pediatricians, family needs to select both doctor in the same clinic, in how many ways this can be done?

Examples for Sec.2.3

- Example3: There are 8 TA's are available, 4 questions need to be marked. How many ways for Prof. To choose 1 TA for each question? How many ways if there are 8 questions?
- Example 4: In a box, there are 10 tennis balls labeled number 1 to 10.
- 1.Randomly choose 4 with replacement
- 2.Choose 4 one by one without replacement
- 3.grab 4 balls in one time
- What is the probability that the ball labelled as number 1 is chosen?

Examples for Sec.2.3

- Example 5: A rental car service facility has 10 foreign cars and 15 domestic cars waiting to be serviced on a particular Sat. morning. Mechanics can only work on 6 of them. If 6 were chosen randomly, what's the probabilty that 3 are domestic 3 are foreign? What's the probabilty that at least 3 domestic cars are chosen?
- Example 6: If a permutation of the word "white" is slelcted at random, find the probability that the permutation
- 1. begins with a consonant
- 2. ends with a vowel
- 3. has the consonant and vowels alternating

Examples for Sec.2.3

•An Economic Department at a state university with five faculty members-Anderson, Box, Cox, Carter, and Davis-must select two of its members to serve on a program review committee. Because the work will be time-consuming, no one is anxious to serve, so it is decided that the representative will be selected by putting five slips of paper in a box, mixing them, and selecting two.

·What is the probability that both Anderson and Box will be selected? (*Hint*: List the equally likely outcomes.)

·What is the probability that at least one of the two members whose name begins with C is selected?

·If the five faculty members have taught for 3, 6, 7, 10, and 14 years, respectively, at the university, what is the probability that the two chosen representatives have at least 15 years' teaching experience at the university?

2.4

Conditional Probability

Example 1

- Two machines produce the same type of products. Machine A produces 8, of which 2 are identified as defective. Machine B produces 10, of which 1 is defective. The sales manager randomly selected 1 out of these 18 for a demonstration.
- What's the probability he selected product from machine A.
- What's the probability that the selected product is defective?
- If the selected product turned to be defective, what's the probability that this product is from machine A?

Conditional Probability

For any two events A and B with $P(B) > 0$, the **conditional probability** of A given that B has occurred is defined by

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

Which can be written:

$$\boxed{P(A \cap B) = P(B) \cdot P(A \mid B)}$$

The Law of Total Probability

If the events $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events. The for any other event B,

$$P(B) = \sum_{i=1}^{k} P(B \mid A_i) P(A_i)$$

Example 2

- Four individuals will donate blood , if only the A+ type blood is desired and only one of these 4 people actually has this type, without knowing their blood type in advance, if we select the donors randomly, what's the probability that at least three individuals must be typed to obtain the desired type?

Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a collection of k mutually exclusive and exhaustive events with $P(A_i) > 0$ for $i = 1, 2, \ldots, k$. Then for any other event B for which $P(B) > 0$ given by

$$\boxed{P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)}}$$

$$j = 1, 2 \ldots, k$$

Ex. A store stocks light bulbs from three suppliers. Suppliers A, B, and C supply 10%, 20%, and 70% of the bulbs respectively. It has been determined that company A's bulbs are 1% defective while company B's are 3% defective and company C's are 4% defective. If a bulb is selected at random and found to be defective, what is the probability that it came from supplier B?

Let D = defective

$$P(B \mid D) = \frac{P(B)P(D \mid B)}{P(A)P(D \mid A) + P(B)P(D \mid B) + P(C)P(D \mid C)}$$

$$= \frac{0.2(0.03)}{0.1(0.01) + 0.2(0.03) + 0.7(0.04)} \approx 0.1714$$

So about 0.17

2.5

Independence

Independent Events

Two event A and B are **independent events** if $P(A \mid B) = P(A).$

Otherwise A and B are dependent.

Properties of independence

- P(B|A)=P(B)
- If A and B are independent, then (1) A' and B (2) A and B' (3) A' and B' are all independent
- Question: A and B are mutually exclusive events, are they independent?

Independent Events

Events A and B are independent events if and only if

$$\boxed{P(A \cap B) = P(A)P(B)}$$

** Note: this generalizes to more than two independent events.

Independent Events

- Events $A_1, \ldots, A_n$ are mutually independent if
- for every k (k=2,3,...,n) and every subset of
- indices $i_1, i_2, \ldots, i_k$,

- $P(A_{i1} \mid A_{i2} \mid \ldots \mid A_{ik}) = P(A_{i1})\, P(A_{i2}) \ldots P(A_{ik})$

Example

- An executive on a business trip must rent a car in each of two different cities. Let A denote the event that the executive is offered a free upgrade in the first city and B represent the analogous event for the second city. Suppose that $P(A) = .3$, $P(B) = .4$, and that A and B are independent events.
- What is the probability that the executive is offered a free upgrade in at least one of the two cities?
- If the executive is offered a free upgrade in at least one of the two cities, what is the probability that such an offer was made only in the first city?
- If the executive is not offered a free upgrade in the first city, what is the probability of not getting a free upgrade in the second city? Explain your reasoning.