Simulating a differential equation with Euler's Method

In these notes I will demonstrate how we can use Euler's Method to approximate the solution to a differential equation. I will also discuss using a spreadsheet program to implement the method. For a general differential equation,

$$\frac{dy}{dt} = f(y,t)\,,$$
$$y(0) = y_0\,,$$

we will approximate the solution at discrete points in time Δt apart. I will define $t_i = i\Delta t$, $i = 0, 1 \ldots$. We let y_1 be the approximation to $y(t_i)$. Euler's method is given by,

$$y_{i+1} = y_i + \Delta t f(y_i, t_i)\,, \quad i = 0, 1, \ldots$$

We will consider the example

$$\frac{dy}{dt} = y^2 t\,,$$
$$y(0) = 1\,.$$

This differential equation has the exact solution,

$$y = \frac{2}{2 - t^2}\,.$$

Note $y' = -\frac{2}{2-t^2)^2}(-2t) = \frac{4t}{(2-t^2)^2} = y^2 t$. I will show how to use Euler's method by hand and then discuss using a spread sheet. We should note that we would expect some trouble as t gets close to $\sqrt{2}$. I will pick $\Delta t = .1$

$$\begin{aligned}
&y_0 = 1\,,\\
&y_1 = y_0 + \Delta t f(y_0, t_0) = 1 + 0.1(1)^2(0) = 1\,,\\
&y_2 = y_1 + \Delta t f(y_1, t_1) = 1 + 0.1(1)^2(.1) = 1.1\,,\\
&y_3 = y_2 + \Delta t f(y_2, t_2) = 1.1 + 0.1(1.1)^2(.2) = 1.342\,,\\
&y_4 = y + 3 + \Delta t f(y_3, t_3) = 1.342 + .1(1.342)^2(.3) = 1.882\,.
\end{aligned}$$

This gives us an approximation at $t = 0, .1, .2, .3, .4$. We could continue on.

Euler's method is not meant to be done by hand though. It is much easier to use a spread sheet program. In this method, we must first make a column for time. So in A1 we type a 0 for t_0. In A2, we type

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=A1+.1
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if $\Delta t = .1$. Then we copy $A2$ and past the contents into the next 10 or so entries in the A column. Now we are ready to finding the approximate solution in the B column. In B1 we type a 1 for y_0. In B2 we type

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=B1+.1*B1^2*A1
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We may then copy the contents of B1 and past it into the B column. We can then try and plot it. I am including a spreadsheet in Excel format in which I have done just this.