Osculating Interpolation

We are given the set of points $\{x_i\}_{i=1}^{q}$. Assume that at each point x_i, we are given the values of the function and up to the m_i^{th} derivative of the function. So we have q function values and $\sum_{i=1}^{q} m_i$ derivative values. Thus, we have $q + \sum_{i=1}^{q} m_i$ different values. The maximum degree of the resulting polynomial is then r where,

$$r = q - 1 + \sum_{i=0}^{q} m_i \,,$$

(recall that n pieces of information results in a polynomial of degree at most $n-1$).

To find the osculating polynomial which goes through each of the data points and matches the derivative values, we write out the data in the following way:

$$(z_1, z_2, \ldots, z_r) = (\underbrace{x_1, x_1, \ldots, x_1}_{m_1+1 \text{ times}}, \underbrace{x_2, x_2, \ldots, x_2}_{m_2+1 \text{ times}}, \ldots, \underbrace{x_q, x_q, \ldots, x_q}_{m_q+1 \text{ times}}) \,.$$

Or in words, if have the function value at x_1 and m_1 derivative values, z_1 through z_{m0} will all equal x_1 and so on till we get to z_r. Now when we wish to find the osculating polynomial we use divided differences with a slight change. We write out the z_i values where we wrote up the x_i's before. We are now going to find the coefficients $f[z_1]$ up to $f[z_1, z_2, \ldots, z_q]$. We define the 0^{th} order divided difference in the same manner as before, $f[z_i] = f(z_i)$. For the higher order divided differences we use,

$$f[z_k, \ldots\; z_i] = \begin{cases} \frac{f[z_{k+1},\ldots,z_i] - f[z_k,\ldots,z_{i-1}]}{z_i - z_k} & z_i \neq z_k \\ \frac{f^{(i-k)}(z_i)}{(i-k)!} & z_i = z_k \end{cases}$$

The recursive relationship is the same as in the previous case if $z_i \neq z_k$. If $z_i = z_k$ then we just use the derivative information. We illustrate with the following example.

We have the fowling table of data,

x_i	$f(x_i)$	$f'(x_i)$	$f''(x_i)$
0	1.0	.5	.25
1.0	1.5	.7	

Now we have 2 points, so $q = 2$. At $x_1 = 0$ we know f' and f'', so $m_1 = 2$. At $x_2 = 1$ we just know f' so $m_2 = 1$. Here $r = 4$ and $(z_1, z_2, z_3, z_4, z_5) = (0, 0, 0, 1, 1)$. We now fill in the table.

0	$f[z_1] = 1$			
0	$f[z_2] = 1$	$f[z_1, z_2] = .5$		
0	$f[z_3] = 1$	$f[z_2, z_3] = .5$	$f[z_1, z_2, z_3] = \frac{.25}{2!} = .125$	
1	$f[z_4] = 1.5$	$f[z_3, z_4] = \frac{f[z_4] - f[z_3]}{z_4 - z_3} = .5$	$f[z_2, z_3, z_4] = \frac{f[z_3,z_4] - f[z_2,z_3]}{z_4 - z_2} = 0$	$f[z_1, z_2, z_3, z_4] = \frac{f[z_2,z_3,z_4] - f[z_1,z_2,z_3]}{z_4 - z_1} = -.125$
1	$f[z_5] = 1.5$	$f[z_4, z_5] = 0.7$	$f[z_3, z_4, z_5] = \frac{f[z_4,z_5] - f[z_3,z_4]}{z_5 - z_3} = .2$	$f[z_2, z_3, z_4, z_5] = \frac{f[z_3,z_4,z_5] - f[z_2,z_3,z_4]}{z_5 - z_2} = .2$

The last entry which doesn't fit is,

$$f[z_1, z_2, z_3, z_4, z_5] = \frac{f[z_2, z_3, z_4, z_5] - f[z_1, z_2, z_3, z_4]}{z_5 - z_1} = .325 \,.$$

The osculating polynomial is then given by,

$$P_4(x) = 1 + .5(x-0) + .125(x-0)^2 - .125(x-0)^3 + .325(x-0)^3(x-1) \,.$$

I checked this polynomial out with Maple and it has all the require properties.