Math 3120 – Differential Equations II
Homework #1 Solutions

1. Find the radius of convergence about the given point, x_0, for which the following differential equations with initial conditions given at x_0 are guaranteed to have a unique solution analytic in the interval $x_0 - R < x < x_0 + R$:
   (a) $y'' + \frac{1}{1+2t}y' + \frac{t}{1-t^2}y = 0$, $t_0 = 0$.
   The singular points are at $t = -\frac{1}{2}$ and $t = \pm 1$. So the radius of convergence is $\frac{1}{2}$
   (b) $(1-9x^2)y'' + 4y' + xy = 0$, $x_0 = 1$
   The singular points are at $x = \pm\frac{1}{3}$, so the radius of convergence is $\frac{2}{3}$.
2. Find the power series solution for the following about the given point. In each case determine the set of values of x for which the series converges and if possible sum the series in closed form.
   (a) $y' = xy$, $y(0) = 5$ about the point $x = 0$.
   We start by assuming a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n ,$$

then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$. We plug into the differential equation to get

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^{n+1} , \tag{1}$$

$$\sum_{n=2}^{\infty} n a_n x^{n-1} + a_1 = \sum_{n=0}^{\infty} a_n x^{n+1} \tag{2}$$

$$\sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} + a_1 = \sum_{n=0}^{\infty} a_n x^{n+1} \tag{3}$$

$$\tag{4}$$

Resulting in the recurrence relation

$$a_{n+2} = \frac{a_n}{n+2}$$

and $a_1 = 0$. All of the odd terms must be zero and the even terms are given by

$$\begin{aligned} a_2 &= \frac{a_0}{2} , \\ a_4 &= \frac{a_2}{4} = \frac{a_0}{(4)(2)} , \\ a_6 &= \frac{a_4}{6} = \frac{a_0}{(8)(3)(2)} , \\ a_8 &= \frac{a_6}{8} = \frac{a_0}{(16)(4)(3)(2)} , \\ &\vdots \end{aligned}$$

The initial condition tells us $a_0 = 5$ and the solutions is given by

$$y = 5\left(1 + \frac{x^2}{2} + \frac{x^4}{2^2 2!} + \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} + \cdots\right)$$

If we look carefully, we can see $y = 5e^{x^2/2}$ and it converges for all x.

(b) $(1+x)y'' + 2y' = 0$ about the point $x = 1$
First we note there is one singular point at $x = -1$, so the radius of convergence is -2. The resulting series solution must then converge for $|x-1| < 2$. We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, then we have

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}\,,$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}\,,$$

To make my life a little easier, I will rewrite the equation as

$$(2 + (x-1))y'' + 2y' = 0$$

We could also redefine the independent variable as $\eta = x - 1$. We plug our series solution into the differential equation to get

$$\sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=1}^{\infty} 2na_n(x-1)^{n-1} = 0\,,$$
$$\sum_{n=1}^{\infty} 2(n+1)(n)a_{n+1}(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} + \sum_{n=1}^{\infty} 2na_n(x-1)^{n-1} = 0\,,$$
$$\sum_{n=2}^{\infty}(2n(n+1)a_{n+1} + (n(n-1) + 2n)a_n)(x-1)^{n-1} + 4a_2 + 2a_1 = 0\,,$$

So after some simplification, we get $a_{n+1} = -\frac{a_n}{2}$ for $n > 1$. So we have the two independent solutions:

$$y_1 = c_1\,,$$
$$y_2 = c_2\left(1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{2^2} - \frac{(x-1)^3}{2^3} + \cdots\right).$$

The general solution can be written as $y = c_1 + \frac{2c_2}{x+1}$.

3. Find the first 5 terms of the power series of the solution to

$$y'' + e^x y = 0\,,\ y(0) = 1\,,\ y'(0) = 0\,.$$

For what values of x would you expect the series to converge.

There are no singular points, so the solution will converge for all x. We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. From the initial conditions, we have $a_0 = 1$ and $a_1 = 0$
We plug our guess into the equation using the Taylor expansion for e^x to get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^n - 2) + \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} a_n x^n\right),$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \Big(a_0 + (a_1 + a_0)x + (a_2 + a_1 + \frac{a_0}{2}) + (a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6})x^3 + (a_4 + a_3 + \frac{a_2}{2} + \frac{a_1}{6} + \frac{a_6}{24})x$$

Matching powers of x, we get

$$\begin{aligned} a_2 &= -\frac{a_0}{2}, \\ a_3 &= -\frac{a_1 + a_0}{6}, \\ a_4 &= -\frac{a_2 + a_1 + \frac{a_0}{2}}{12}, \\ a_5 &= -\frac{a_3 + a_2 + \frac{a_1}{2} + \frac{a_0}{6}}{20}. \end{aligned}$$

Or

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 0, \\ a_2 &= -\frac{1}{2}, \\ a_3 &= -\frac{1}{6}, \\ a_4 &= 0, \\ a_5 &= \frac{1}{40}. \end{aligned}$$

The series is then,

$$y = 1 - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{40}.$$

4. Consider the following differential equation

$$(1 - x^2)y'' - xy' + \mu^2 y = 0.$$

(a) Find and classify all singular points.
The singular points are at $x = \pm 1$ and they are both regular.

(b) Find the recurrence relationship for a series expansion about $x = 0$.
Since 0 is a regular point, we may assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. If we

plug this into the differential equation, we get

$$(1-x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + \mu^2 \sum_{n=0}^{\infty} a_n x^n \,,$$

$$\sum_{n=2}^{\infty}\left(a_n n(n-1)x^{n-2} - a_n n(n-1)x^n\right) - \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=0}^{\infty} \mu^2 a_n x^n \,,$$

$$\sum_{n=0}^{\infty}(a_{n+2}(n+2)(n+1) - a_n(n^2-\mu^2))x^n = 0 \,,$$

If you carefully write out all the terms, you will see that the above is correct. The recurrence relation is then given by

$$a_{n+2} = \frac{n^2-\mu^2}{(n+2)(n+1)} a_n$$

Since this is a second order recursion relation, we can get two independent solutions by first setting $a_0 = 1$ and $a_1 = 0$ and then $a_0 = 0$ and $a_1 = 1$.

(c) Show that if μ is an integer, the series solutions is just a polynomial of degree μ.
If we set μ to be an odd integer, and we set $a_0 = 0$ and $a_1 = 1$. If μ is even, we set $a_0 = 1$ and $a_1 = 0$. In either case, we will get $a_m = 0$ if $m > n$.

(d) The polynomial solutions to this equation are referred to as Chebyshev polynomials of the first kind. Find the first 3 such polynomials.
The first two polynomials are clearly $T_0 = 1$ and $T_1 = x$. To get the third, we set $\mu = 2$ and $a_0 = 1$, so $a_2 = \frac{-4}{2}a_0 = -2$ and then $T_2 = -2x^2 + 1$. I didn't ask for it, but we can get the third. In this case $a_1 = 1$ and $a_3 = \frac{1-9}{(3)(2)}a_1 = -\frac{4}{3}$ and $T_3 = -\frac{4}{3}x^3 + 1$. These are not the standard Chebyshev polynomials, but they are solutions to the differential equation.

5. Find the general power series solution about $x = 0$ for the differential equation

$$xy'' + 2y' + xy = 0 \,.$$

First $x = 0$ is a singular point, so we expect the solution to be in the form $y = \sum_{n=0}^{\infty} a_n x^{n+4}$. To find r, we look at the indicial equation. Here $p_0 = \lim_{x\to 0} xp(x) = 2$ and $q_0 = \lim_{x\to 0} x^2 q(x) = 0$. So the indicial equation is $r(r-1) + 2r = 0$, so the two roots are $r = 0$ and $r = -1$. Since the two roots differ by an integer, we may be in for some difficulties. If there is a problem, it is with the smaller of the roots $r = -1$. Let's begin with case $r = -1$ to see what happens.

$$y = \sum_{n=0}^{\infty} a_n x^{n-1} \,,$$

$$y' = \sum_{n=0}^{\infty}(n-1)a_n x^{n-2} \,,$$

$$y'' = \sum_{n=0}^{\infty}(n-1)(n-2)a_n x^{n-3} \,.$$

We substitute into the differential equation to get,

$$\sum_{n=0}(n-1)(n-2)a_n x^{n-2} + \sum_{n=0}^{\infty} 2(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0\,,$$
$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n\,,$$

Note that the first two terms of the first sum are 0. This is a result of the roots differing by an integer. Generically, this would result in something like $a_1 = \frac{C}{0}a_0$, but we are in luck here and we should be able to find a solution. We continue,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n\,,$$
$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0\,,$$
$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + a_n\right)x^n = 0\,,$$

So we have the recurrence relation,

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}\,.$$

We are really lucky here, since we may use this relation to get the series solution for both roots. We get

$$y = c_1\left(x^{-1} - \frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots\right) + c_2\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\right)\,.$$

It is then clear that the solutions are given by

$$y = \frac{c_1\cos(x) + c_2\sin(x)}{x}\,.$$

6. Find the first 4 terms in the general power series solution to

$$y'' - xy' - y = 5\sqrt{x}$$

Hint: Find a particular solution of the form $x^r(b_0 + b_1 x + b_2 x^2 + \cdots)$ then solve $y'' - xy' - y = 0$ as we have done previously.
I will begin by finding the general solution to the homogeneous problem

$$y'' - xy' - y = 0\,.$$

The point $x = 0$ is a regular point, so we seek a solution in the form $y = \sum_{n=0}^{\infty} a_n x^n$. We substitute into the equation

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,,$$

$$\sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + 2a_2 - \sum_{n=1}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0\,,$$

$$\sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} - n a_n - a_n\right) x^n + 2a_2 - a_0 = 0\,.$$

We can then get the recurrence relation,

$$a_{n+2} = \frac{n+1}{(n+2)(n+1)} a_n = \frac{a_n}{n+2}$$

We can then construct two independent solutions,

$$y_1 = c_1 \left(1 + \frac{x^2}{2} + \frac{x^4}{4 \cdot 2} + \frac{x^6}{6 \cdot 4 \cdot 2} + \cdots \right)\,,$$

$$y_2 = c_2 \left(x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 3} + \frac{x^7}{7 \cdot 5 \cdot 3} + \cdots \right)\,.$$

Now we look for a particular solution to the problem. We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and plug it into the equation to get,

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} + \sum_{n=0}^{\infty} a_n (n+4) x^{n+4} - \sum_{n=0}^{\infty} a_n x^{n+r} = 5x^{\frac{1}{2}}\,,$$

$$\left(\sum_{n=0}^{\infty} a_{n+2}(n+2+r) x^{n+4}\right) + a_0 r(r-1)x^{r-2} + a_1 r(r+1) - \sum_{n=0}^{\infty} a_n (n+r) x^{n+4} - \sum_{n=0}^{\infty} a_n x^{n+r} = 5x^{\frac{1}{2}}\,,$$

$$\left(\sum_{n=0}^{\infty} \left(a_{n+2}(n+r+2)(n+r+1) - a_n(n+r+1)\right) x^{n+r}\right) + a_1 r(r+1)x^{r-1} + a_0 r(r-1)x^{r-2} = 5x^{\frac{1}{2}}\,,$$

We have the recursion relation

$$a_{n+2} = \frac{a_n}{n+r+2}\,.$$

We have two choices, we can set $r = \frac{5}{2}$, $a_0 = \frac{4}{3}$ and $a_1 = 0$ or we cans set $a_0 = 0$, $r = \frac{3}{2}$ and $a_1 = \frac{4}{3}$. In either case, we get the same answer,

$$y = y_1 + y_2 + \frac{4}{3} x^{5/2} \left(1 + \frac{2x^2}{9} + \frac{4x^4}{9 \cdot 13} + \frac{8x^6}{9 \cdot 13 \cdot 17} + \cdots \right)\,.$$