

Math 3120 – Differential Equations II

Homework #2 Due Monday February 8

Note: Two Pages.

1. Consider the following differential equation:

$$\begin{aligned}u_t(x, t) &= u_{xx}(x, t), \quad 0 < x < 1, \quad t > 0 \\u(0, t) &= u(1, t) = 0, \\u(x, 0) &= f(x).\end{aligned}$$

Find a Fourier series solution in the following cases:

- (a) $f(x) = \sin(8x)$.
(b) $f(x) = x(1 - x)$
2. Let $u(x, t)$ be the temperature of a thin insulated wire of length l . The energy contained within the bar is given by

$$E(t) = c_0 A \int_0^l u(x, t) dx,$$

where c_0 is the heat capacity and A is the cross-sectional area. If both ends of the bar are insulated, we would expect the energy to a constant value. Use the heat equation to show this is in fact the case.

3. Consider a bar of length l whose left end (at $x = 0$) is kept at zero degrees and whose right end (at $x = l$) is insulated.
- (a) State the differential equation and boundary conditions appropriate for this situation.
(b) Suppose we apply separation of variables and look for a solution of the form $u(x, t) = X(x)T(t)$. What are the separation equations for X and T ? What are the boundary conditions that $X(x)$ must satisfy?
(c) What are the allowable values for the separation constant?
4. Consider the wave equation on the real line:

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \quad (1a)$$

$$u(x, 0) = f(x), \quad (1b)$$

$$u_t(x, 0) = g(x). \quad (1c)$$

$$(1d)$$

In class, I showed that if $y(x, t) = F(x - ct) + G(x + ct)$, for any twice differentiable functions F and G , then y solves the wave equation (1a). And that the solution to the above problem (1) is given by,

$$u(x, t) = \frac{1}{2}(f(x + ct) + f(x - ct)) - \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

Show that this is the solution by applying the boundary conditions to the function $y = F(x - ct) + G(x + ct)$ and solving for F and G .

5. Consider the following Laplace's equation:

$$\begin{aligned}u_{xx}(x, y) + u_{yy}(x, y) &= 0, & 0 < x < 1, & \quad 0 < y < 1, \\u(1, y) &= 1, & 0 \leq y < 1, \\u(x, 1) &= 1, & 0 < x \leq 1, \\u(0, y) &= y, & 0 < y \leq 1, \\u(x, 0) &= x, & 0 \leq x < 1.\end{aligned}$$

- (a) Determine a function $v(x, y)$ that has the prescribed boundary conditions at the four corners.
- (b) Form $U(x, y) = u(x, y) - v(x, y)$. Then formulate and solve the corresponding problem for $U(x, y)$ in terms of a Fourier series.