Math 3120 – Differential Equations II

Homework #2 Solutions

1. Consider the following differential equation:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t)\,, \quad 0<x<1\,, \quad t>0 \\ u(0,t) &= u(1,t) = 0\,, \\ u(x,0) &= f(x)\,. \end{aligned}$$

Find a Fourier series solution in the following cases:
Since the boundary conditions are 0 at both ends, the series must be of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 t}\sin(n\pi x)\,.$$

(a) $f(x) = \sin(8x)$. In this case,

$$\begin{aligned} a_n &= 2\int_0^1 \sin(8x)\sin(n\pi x)\,dx\,, \\ &= \frac{2n\pi\sin(8)(-1)^n}{64-n^2\pi^2}\,. \end{aligned}$$

(b) $f(x) = x(1-x)$ Here,

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\sin(n*pi*x)\,, \\ &= \frac{4-4(-1)^n}{n^3\pi^3}\,. \end{aligned}$$

2. Let $u(x,t)$ be the temperature of a thin insulated wire of length l. The energy contained with in the bar is given by

$$E(t) = c_0 A\int_0^l u(x,t)\,dx\,,$$

where c_0 is the heat capacity and A is the cross-sectional area. If both ends of the bar are insulated, we would expect the energy to a constant value. Use the heat equation to show this is in fact the case.
To show that the energy doesn't change, we just need to show its time derivative is 0.

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\left(c_0 A\int_0^l u(x,t)\,dx\right)\,, \\ &= c_0 A\int_0^l u_t(x,t)\,dx\,, \\ &= c_0 A\int_0^l k u_{xx}(x,t)\,dx\,, \\ &= c_0 A\left(k u_x\big|_0^l\right)\,, \\ &= 0\,. \end{aligned}$$

3. Consider a bar of length l whose left end (at $x = 0$) is kept at zero degrees and whose right end (at $x = l$) is insulated.

   (a) State the differential equation and boundary conditions appropriate for this situation.

$$\begin{aligned} u_t &= k u_{xx} \,, \\ u(0,t) &= 0 \,, \\ u_x(l,t) &= 0 \,, \\ u(x,0) &= u_0(x) \,. \end{aligned}$$

   (b) Suppose we apply separation of variables and look for a solution of the form $u(x,t) = X(x)T(t)$. What are the separation equations for X and T? What are the boundary conditions that $X(x)$ must satisfy?

$$\begin{aligned} \frac{T'}{kT} &= \frac{X''}{X} = -\alpha^2 \,, \\ X(0) &= 0 \,, \\ X'(l) &= 0 \,. \end{aligned}$$

   (c) What are the allowable values for the separation constant?
   Since the separation constant must be negative for the solution to decay in time, we have the following solution for X:

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x) \,.$$

   Since $X(0) = 0$, we have that $A = 0$. Since $X'(l) = 0$, we have $\alpha = \frac{(2n-1)\pi}{2l}$ for $n = 1, 2, \ldots$. Note that in this case, the constant solution is not a nontrivial solution due to the boundary condition on the left end.

4. Consider the wave equation on the real line:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \,, \quad -\infty < x < \infty \,, \quad t > 0 \,, \qquad &(1a) \\ u(x,0) &= f(x) \,, &(1b) \\ u_t(x,0) &= g(x) \,. &(1c) \\ & &(1d) \end{aligned}$$

In class, I showed that if $y(x,t) = F(x-ct) + G(x+ct)$, for any twice differentiable functions F and G, then y solves the wave equation (1a). And that the solution to the above problem (1) is given by,

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct) - \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \,.$$

Show that this is the solution by applying the boundary conditions to the function $y = F(x-ct) + G(x+ct)$ and solving for F and G.
The two conditions give the following:

$$\begin{aligned} u(x,0) &= F(x) + G(x) = f(x) \,, \qquad &(2) \\ u_t(x,0 &= -cF'(x) + cG'(x) = g(x) \,, \,. &(3) \\ & &(4) \end{aligned}$$

We take the derivative of (2) and multiply by c and add the result to(3) to get,

$$2cG'(x) = cf'(x) + g(x) \,.$$

We can integrate this directly to get

$$\begin{aligned} G(x) &= \int_{x_0}^{x} \left(\frac{1}{2} f'(s) + \frac{1}{2c} g(s) \right) \, ds \,, \\ &= \frac{1}{2}(f(x) - f(x_0)) + \frac{1}{2c} \int_{x_0}^{x} g(s) \, ds \,. \end{aligned}$$

Now using (2) we have

$$\begin{aligned} F(x) &= f(x) - G(x) \,, \\ &= \frac{1}{2}(f(x) + f(x_0)) - \frac{1}{2c} \int_{x_0}^{x} g(s) \, ds \,. \end{aligned}$$

So the solution to the differential equation is given by

$$\begin{aligned} u(x,t) &= F(x - ct) + G(x + ct) \,, \\ &= \frac{1}{2}(f(x - ct) + f(x_0)) - \frac{1}{2c} \int_{x_0}^{x-ct} g(s) \, ds + \frac{1}{2}(f(x + ct) - f(x_0)) + \frac{1}{2c} \int_{x_0}^{x+ct} g(s) \, ds \,, \\ &= \frac{1}{2}(f(x + ct) + f(x - ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) \, ds \,. \end{aligned}$$

5. Consider the following Laplace's equation:

$$\begin{aligned} u_{xx}(x,y) + u_{yy}(x,y) &= 0 \,, \quad 0 < x < 1 \,, \quad 0 < y < 1 \,, \\ u(1,y) &= 1 \,, \quad 0 \le y < 1 \,, \\ u(x,1) &= 1 \,, \quad 0 < x \le 1 \,, \\ u(0,y) &= y \,, \quad 0 < y \le 1 \,, \\ u(x,0) &= x \,, \quad 0 \le x < 1 \,. \end{aligned}$$

   (a) Determine a function $v(x,y)$ that has the prescribed boundary conditions at the four corners.
   We let $v(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$ and then get:

$$\begin{aligned} v(0,0) = 0 &\implies \alpha_1 = 0 \,, \\ v(0,1) = 1 &\implies \alpha_3 = 1 \,, \\ v(1,0) = 1 &\implies \alpha_2 = 1 \,, \\ v(1,1) = 1 &\implies 2 + \alpha_4 = 1 \,, \\ &\qquad \alpha_4 = -1 \,. \end{aligned}$$

   So,

$$v(x,y) = x + y - xy$$

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(b) Form $U(x,y) = u(x,y)) - v(x,y)$. Then formulate and solve the corresponding problem for $U(x,y)$ in terms of a Fourier series.
We let $U(x,y) = u(x,y) - v(x,y)$ then the boundary conditions for U are given by,

$$\begin{aligned} U(x,0) &= u(x,0) - v(x,0) = x - x = 0\,, & (5)\\ U(0,y) &= u(0,y) - v(0,y) = y - y = 0\,, & (6)\\ U(x,1) &= u(x,y) - v(x,1) = 1 - (x + 1 - x) = 0\,, & (7)\\ U(1,y) &= u(1,y) - v(1,y) = 1 - (1 + y - y) = 0\,. & (8)\\ & & (9) \end{aligned}$$

Since U is zero on all the boundaries, then U is identically 0 and thus $u = v$ is the solution to the original differential equation.