Math 3120 – Differential Equations II

Homework #3 Due Friday March 19

1. The equation $y'' = 1 - (1+y)^{3/2}$ models the motion of a bobbing parabolic trough.
   (a) Write the equation as a first order system in x and y.
   (b) Show the system has a single equilibrium at $(x, y) = (0, 0)$.
   (c) Determine the eigenvalues of the linearized system. What do they tell you about the behaviour of the solutions.
   (d) Find a quantity $H(x, y)$ which is constant along solutions to the equation.
   (e) Use the Maple implicit plot function to plot the trajectories $H(x, y) = .5$ and $H(x, y) = 1$. Say something about the stability of the origin. Since the choice for H is not unique, yours may not work. If you aren't getting a graph try $H(x, y) = -.5$ and $H(x, y) = .5$ instead.
2. Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}. \tag{1}$$

   (a) Show that changing to poloar coordingates $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$ results in the system

$$\begin{aligned} r' &= r(a_{11}\cos^2(\theta) + a_{22}\sin^2(\theta) + (a_{12}+a_{21})\cos(\theta)\sin(\theta)) + g_1\cos(\theta) + g_2\sin(\theta)\,, \\ \theta' &= (a_{21}\cos^2(\theta) - a_{12}\sin^2(\theta) + (a_{22}-a_{11})\sin(\theta)\cos(\theta)) + \frac{-g_1\sin(\theta) + g_2\cos(\theta)}{r}\,, \end{aligned}$$

   where $g_1 = g_1(r\cos(\theta), r\sin(\theta))$ and $g_2 = g_2(r\cos(\theta), r\sin(\theta))$.

   (b) Show that if

$$a_{11} = a_{22} \quad a_{21} = -a_{12} \quad g_1(x,y) = xh(\sqrt{x^2+y^2}) \quad g_2 = yh(\sqrt{(x^2+y^2})$$

   for some function h, then the polar equations uncouple to the separable differential equations

$$\begin{aligned} r' &= a_{11}r + rh(r)\,, \\ \theta' &= a_{21}\,. \end{aligned}$$

   (c) Use the above to solve

$$\begin{aligned} x' &= y + \alpha x(x^2+y^2)\,, \\ y' &= -x + \alpha y(x^2+y^2)\,. \end{aligned}$$

3. Consider a disease spreading through a fixed population. Let $s(t)$ represent the fraction of the population which is healthy and susceptible to the disease. Let $i(t)$ represent the fraction

infected and $r(t)$ the fraction which have recovered and are still immune to the disease. We model the population dynamics as follows:

$$\begin{aligned} s' &= -\alpha s i + \gamma r \,, \\ i' &= \alpha s i - \beta i \,, \\ r' &= \beta i - \gamma r \,. \end{aligned}$$

(a) Show that the total size of the community $N(t) = s(t) + i(t) + r(t)$ is constant in time.

(b) Use the fact that the population size is constant to reduce the system to a system of two differential equations for $s(t)$ and $i(t)$.

(c) Set $\alpha = \beta = \gamma = 1$ and $N = 9$ and determine all the equilibria of the system.

(d) Use the eigenvalues of the linearized system to classify the equilibria as in the table in Figure 7.3.9 on page 506 of the text.

(e) How would the system be modified if the immunity was permanent?