Math 3120 – Differential Equations II

Homework #3 Solutions

1. The equation $y'' = 1 - (1+y)^{3/2}$ models the motion of a bobbing parabolic trough.

   (a) Write the equation as a first order system in x and y.

$$y' = x\,,$$
$$x' = 1 - (1-y)^{3/2}\,,$$

   (b) Show the system has a single equilibrium at $(x, y) = (0, 0)$.
   Clearly we require $x = 0$. If we set

$$\begin{aligned} 1 - (1-y)^{3/2} &= 1\,, \\ (1-y)^{3/2} &= 1\,, \\ 1 - y &= 1\,, \\ y &= 0\,. \end{aligned}$$

   (c) Determine the eigenvalues of the linearized system. What do they tell you about the behaviour of the solutions.
   The Jacobian is given by

$$\begin{pmatrix} 0 & 1 \\ \frac{3}{2}(1-y)^{1/2} & 0 \end{pmatrix}$$

   If we set $(x, y) = (0, 0)$, we get the matrix

$$\begin{pmatrix} 0 & 1 \\ \frac{3}{2} & 0 \end{pmatrix}$$

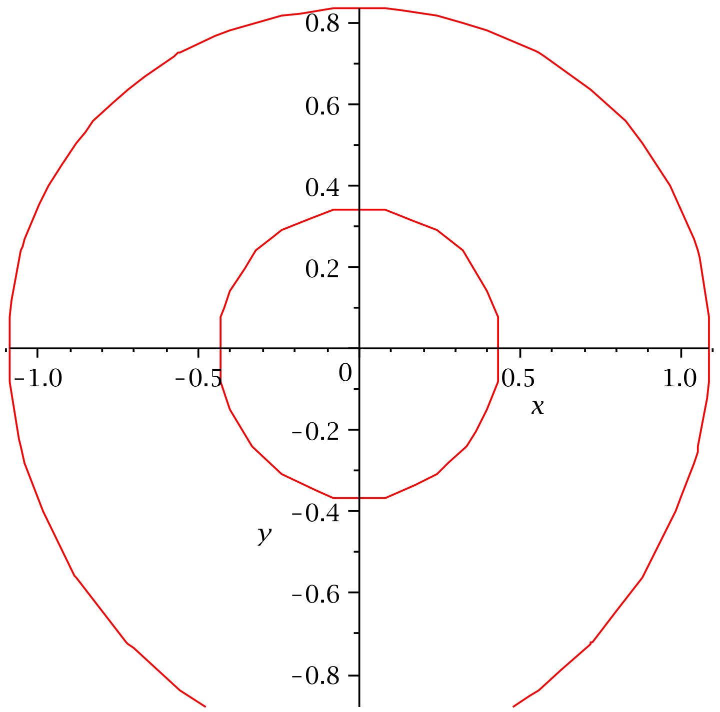
   The eigenvalues of this matrix are $\sqrt{\frac{3}{2}}i$. Since the eigenvalues have zero real part, we can't say anything about the stability of the origin.

   (d) Find a quantity $H(x, y)$ which is constant along solutions to the equation.
   On such $H(x, y)$ will satisfy

$$\frac{\partial H}{\partial x} = x\,,$$
$$\frac{\partial H}{\partial y} = -(1-(1-y)^{3/2}\,.$$

   One possible solution is $H = \frac{1}{2}x^2 - y + \frac{2}{5}(1-y)^{5/2}$.

   (e) Use the Maple implicit plot function to plot the trajectories $H(x, y) = .5$ and $H(x, y) = 1$. Say something about the stability of the origin. Since the choice for H is not unique, yours may not work. If you aren't getting a graph try $H(x, y) = -.5$ and $H(x, y) = .5$ instead.
   Here is the plot I got from maple.



As you can see the solutions are periodic orbits near the origin. Periodic orbits are hard to find and can't be show to exist just by looking at the eigenvalues of the linearized problem. You have to resort to something like a Hamiltonian function.

2. Consider the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix} . \tag{1}$$

(a) Show that changing to poloar coordingates $x(t) = r(t)\cos(\theta(t))$, $y(t) = r(t)\sin(\theta(t))$ results in the system

$$r' = r(a_{11}\cos^2(\theta) + a_{22}\sin^2(\theta) + (a_{12}+a_{21})\cos(\theta)\sin(\theta)) + g_1\cos(\theta) + g_2\sin(\theta)\,,$$
$$\theta' = (a_{21}\cos^2(\theta) - a_{12}\sin^2(\theta) + (a_{22}-a_{11})\sin(\theta)\cos(\theta)) + \frac{-g_1\sin(\theta)+g_2\cos(\theta)}{r}\,,$$

where $g_1 = g_1(r\cos(\theta), r\sin(\theta))$ and $g_2 = g_2(r\cos(\theta), r\sin(\theta))$.
We substitute in to get

$$\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} .$$

Now we just need to find the inverse of the matrix on the right.

$$\left(\begin{array}{cc|cc} \cos(\theta) & -r\sin(\theta) & 1 & 0 \\ \sin(\theta) & r\cos(\theta) & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \cos(\theta) & -r\sin(\theta) & 1 & 0 \\ 0 & r & -\sin(\theta) & \cos(\theta) \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \cos(\theta) & 0 & 1-\sin^2(\theta) & \sin(theta)\cos(\theta) \\ 0 & r & -\sin(\theta) & \cos(\theta) \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 1 & -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{array}\right)$$

So the system is given by

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{r} & \frac{\cos(\theta)}{r} \end{pmatrix} \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right].$$

If we multiply this out, we have the result.

(b) Show that if

$$a_{11} = a_{22} \quad a_{21} = -a_{12} \quad g_1(x,y) = xh(\sqrt{x^2+y^2}) \quad g_2 = yh(\sqrt{(x^2+y^2})$$

for some function h, then the polar equations uncouple to the separable differential equations

$$\begin{aligned} r' &= a_{11}r + rh(r)\,, \\ \theta' &= a_{21}\,. \end{aligned}$$

First if g_1 and g_2 are as given, then

$$\begin{aligned} g_1\cos(\theta) + g_2\sin(\theta) &= x\cos(\theta)h(r) + y\sin(\theta)h(r)\,, \\ &= r\cos^2(\theta)h(r) + r\sin^2(\theta)h(r)\,, \\ &= rh(r)\,. \\ \frac{-g1\sin(\theta) + g_2\cos(\theta)}{r} &= \frac{-x\sin(\theta)h(r) + y\cos(\theta)h(r)}{r}\,, \\ &= \frac{-r\cos(\theta)\sin(\theta)h(r) + r\cos(\theta)\sin(\theta)h(r)}{r}\,, \\ &= 0\,. \end{aligned}$$

The rest is clear.

(c) Use the above to solve

$$\begin{aligned} x' &= y + \alpha x(x^2+y^2)\,, \\ y' &= -x + \alpha y(x^2+y^2)\,. \end{aligned}$$

Here $h(r) = \alpha r^2$, $a_{11} = a_{22} = 0$ and $a_{12} = -a_{21} = 1$. The equation in polar co-ordinates is then

$$\dot{r} = \alpha r^3 \,.$$
$$\dot{\theta} = -1 \,.$$

So the general solution is

$$r(t) = \frac{r_0}{\sqrt{1 - 2r_0^2 \alpha t}} \,,$$
$$\theta(t) = -t + \theta_0 \,.$$

3. Consider a disease spreading through a fixed population. Let $s(t)$ represent the fraction of the population which is healthy and susceptible to the disease. Let $i(t)$ represent the fraction infected and $r(t)$ the fraction which have recovered and are still immune to the disease. We model the population dynamics as follows:

$$\begin{aligned} s' &= -\alpha s i + \gamma r \,, \\ i' &= \alpha s i - \beta i \,, \\ r' &= \beta i - \gamma r \,. \end{aligned}$$

(a) Show that the total size of the community $N(t) = s(t) + i(t) + r(t)$ is constant in time.

$$\begin{aligned} \frac{dN}{dt} &= \frac{ds}{dt} + \frac{di}{dt} + \frac{dr}{dt} \,, \\ &= -\alpha s i + \gamma r + \alpha s i - \beta i + \beta i - \gamma r \,, \\ &= 0 \,. \end{aligned}$$

So the total population is constant.

(b) Use the fact that the population size is constant to reduce the system to a system of two differential equations for $s(t)$ and $i(t)$.
Since the population is constant, we can replace r by $N - s - i$ to get the system

$$\begin{aligned} s' &= -\alpha s i + \gamma (N - s - i) \,, \\ i' &= \alpha s i - \beta i \,, \end{aligned}$$

(c) Set $\alpha = \beta = \gamma = 1$ and $N = 9$ and determine all the equilibria of the system.
We need to solve

$$\begin{aligned} -si + (9 - s - i) &= 0 \,, \\ si - i &= 0 \,. \end{aligned}$$

We have one solution with $i = 0$ and $s = 9$ (disease free) and the other has $s = 1$, $i = 4$ (this is the endemic solution, also note $r = 4$ here).

(d) Use the eigenvalues of the linearized system to classify the equilibria as in the table in Figure 7.3.9 on page 506 of the text.

The Jacobian is given by

$$\begin{pmatrix} -i-1 & -s-1 \\ i & s-1 \end{pmatrix}$$

At $(9,0)$, the Jacobian is given by

$$\begin{pmatrix} -2 & -10 \\ 0 & 8 \end{pmatrix}$$

The characteristic equation is $(-2-\lambda)(8-\lambda)$ so $(9,0)$ is a saddle point.
At $(4,1)$ the Jacobian is given by

$$\begin{pmatrix} -5 & -2 \\ 4 & 0 \end{pmatrix}$$

The characteristic equation is $-\lambda(-5-\lambda)+8=0$, or $\lambda^2+5\lambda+8=0$. The eigenvalues are $\lambda=-\frac{5}{2}\pm\frac{\sqrt{7}}{2}i$. So $(4,1)$ is a spiral point. Since the real part of the eigenvalues are negative, it is a stable spiral point.

(e) How would the system be modified if the immunity was permanent?
If immunity becomes permanent, then the term $-\gamma r$ ins the r' equation and the term γr in the s' equation would be gone (these are the terms responsible for loss of immunity and the subsequent becoming susceptible). The new system is then

$$\begin{aligned} s' &= -\alpha s i \\ i' &= \alpha s i - \beta i\,, \\ r' &= \beta i \end{aligned}$$

In this new system there are infinitely many equilibria. We just set $i=0$, and divide the remaining population among s and r.