Math 3120 Practise Test

1. Find the first 3 non-zero terms of the power series solution about $x = 0$ in two independent solutions of the equation

$$(1 + x^3)y'' + x^4 y = 0\,.$$

Comment on the radius of convergence.

2. Consider the equation with $x = 0$ as a regular singular point.

$$\sin(x)y'' - y' + y = 0\,.$$

   (a) Verify that $x = 0$ is a singular point.

   (b) Find and solve the indicial equation.

   (c) Find the first 3 terms of the series solution corresponding to the larger root of the indicial equation.

3. Consider the equation for the temperature of a thin bar with partially insulated endpoints.

$$u_t = u_{xx}\,, \quad 0 < x < 1\,, \quad t > 0\,,$$
$$u'(0) = -u(0)\,, \quad u_x(1,t) = 2u(1,t)\,,$$
$$u(x,0) = g(x)\,.$$

   (a) Assume a solution in the form $u(x,t) = X(x)T(t)$ and find the differential equations that X and T must satisfy.

   (b) Find the relationship the eigenvalues must satisfy.

4. The displacement $y(x,t)$ of a vibrating string under the influence of gravity must satisfy

$$y_{tt} = y_{xx} - g\,.$$

If the endpoints are fixed then $y(0,t) = y(L,t) = 0$.

   (a) If the string is stationary ($y_{tt} = 0$), find the displacement of the string, $\phi(x)$.

   (b) If the string is released with no initial velocity or displacement then $y(x,0) = y_t(x,0) = 0$. Let $v(x,t) = y(x,t) - \phi(x)$. What equation must v satisfy?

   (c) Solve for $v(x,t)$ and thus $y(x,t)$.

5. Consider the eigenvalue problem

$$u'' + u' + u = -\lambda u\,, \quad 0 < x < 1\,,$$
$$u(0) = 0\,,$$
$$u(1) + 2u(1) = 0\,.$$

(a) Recast the problem in the form as a Sturm-Liouville problem.

(b) Find the eigenvalues, λ_i and eigenfunctions $\phi_i(x)$.

(c) Given a reasonable function $f(x)$ express it as an eigenfunction expansion $f = \sum_{n=0}^{\infty} c_i \phi_i(x)$. The formula for c_i should be in terms of f and know quantities.

6. Denote the population of a desirable species by $x(t)$ and an undesirable competing species by $y(t)$. If resources are used to reduce the population of y, then we can model the dynamics of the populations by

$$x' = r(1 - \alpha x - \beta y)x \,,$$
$$y' = r(1 - \alpha y - \beta x)y - \mu y \,.$$

Assume $\alpha > \beta > 0$ and r and μ are both greater than 0.

(a) Determine the four equilibria of the system.

(b) Show that if μ is large enough then there are only two viable equilibria (in the first quadrant since population can't be negative).

(c) Assume μ is large enough so that there are only two physically viable equilibria as above. Compute the linearized stability of both.

7. The 2nd order midpoint Runge-Kutta method for approximating $y' = f(y,t)$ is given by

$$y_{n+1} = y_n + hf(y_n + \frac{h}{2}f(y_n, t_n), t_n + \frac{h}{2})$$

where h is the step size, $t_n = nh$ and y_0 is given by the initial condition.

(a) Apply the method to $y' = \lambda y$. Give the equation for y_{n+1} in terms of y_n

(b) What must $h\lambda$ satisfy to ensure $\left|\frac{y_{n+1}}{y_n}\right| < 1$.

In addition to questions like the above the test will also have a page of short answer type questions.

A formula sheet is allowed.

I will be posting additional questions soon.