Math 3120 Practise Test

1. Find the first 3 non-zero terms of the power series solution about $x = 0$ in two independent solutions of the equation

$$(1 + x^3)y'' + x^4 y = 0\,.$$

Comment on the radius of convergence.

Since $x = 0$ is a regular point, we assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n\,.$$

We substitute into the differential equation to get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+4}\,.$$

The lowest power of x is $n-2$, so we adjust the summation indices's make all the powers of x, $n-2$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=5}^{\infty} a_{n-3}(n-3)(n-4)x^{n-2} + \sum_{n=6}^{\infty} a_{n-6}x^{n-2}\,.$$

Now we collect what we can into the summation.

$$\sum_{n=6}^{\infty}(a_n n(n-1)+a_{n-3}(n-3)(n-4)+a_{n-6})x^{n-2}+2a_2+6a_3x+12a_4x^2+20a_5x^3+2a_2x^3\,.$$

We can see right away that $a_2 = a_3 = a_4 = a_5 = 0$ and we have the recurrence relation

$$a_n = -\frac{a_{n-3}(n-3)(n-4) + a_{n-6}}{n(n-1)}\,.$$

For the first independent solution, we set $a_0 = 1$ and $a_1 = 0$. Then

$$a_6 = -\frac{a_0}{30} = -\frac{1}{30}\,,$$
$$a_7 = 0\,,$$
$$a_8 = 0\,,$$
$$a_9 = -\frac{a_6(30)}{72} = \frac{1}{72}\,.$$

So one independent solution is given by

$$y_1 = 1 - \frac{1}{30}x^6 + \frac{1}{72}x^9 + \cdots\,.$$

For the other independent solution we set $a_0 = 0$ and $a_1 = 0$ then we have

$$\begin{aligned}
a_6 &= 0\,, \\
a_7 &= -\frac{a_1}{42} = -\frac{1}{42}\,, \\
a_8 &= 0\,, \\
a_9 &= 0\,, a_{10} &= -\frac{a_7(42)}{90} = \frac{1}{90}\,.
\end{aligned}$$

and the second independent solution is given by

$$y_2 = x - \frac{1}{42}x^7 + \frac{1}{90}x^{10} + \cdots\,.$$

2. Consider the equation with $x = 0$ as a regular singular point.

$$\sin(x)y'' - y' + y = 0\,.$$

(a) Verify that $x = 0$ is a singular point.
In this case, $p(x) = \frac{1}{\sin(x)} = q(x)$ and $\lim_{x\to 0} p(x) = \lim_{x\to 0} q(x) = \infty$. However, $\lim_{x\to 0} xp(x) = -1$ and $\lim_{x\to 0} x^2q(x) = 0$, so $x = 0$ is a regular singular point.

(b) Find and solve the indicial equation.
The indicial equation is given by $r(r-1) - r = 0$ $r(r-2) = 0$. The roots are $r = 0$ and $r = 2$.

(c) Find the first 3 terms of the series solution corresponding to the larger root of the indicial equation.
We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^{n+2}$ and sub into the differential equation.

$$(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots)\sum_{n=0}^{\infty} a_n(n+2)(n+1)x^n - \sum_{n=0}^{\infty} a_n(n+2)x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2}\,,$$

$$(2a_0 - 2a_0)x + (6a_1 - 3a_1 + a_0)x^2 + (a_2(12) - a_0\frac{2}{6} - a_2(4) + a_1)x^3 + \cdots\,.$$

We may choose any value for a_0 so we set it to 1 then we have

$$\begin{aligned}
a_1 &= -\frac{a_0}{3} = -\frac{1}{3}\,, \\
a_2 &= \frac{a_0\frac{2}{6} - a_1}{8} = \frac{1}{12}\,.
\end{aligned}$$

The first three terms are then

$$y = 1 - \frac{1}{3}x + \frac{1}{12}x^2 + \cdots$$

3. Consider the equation for the temperature of a thin bar with partially insulated endpoints.

$$\begin{gathered} u_t = u_{xx}\,, \quad 0 < x < 1\,, \quad t > 0\,, \\ u'(0) = -u(0)\,, \quad u_x(1,t) = 2u(1,t)\,, \\ u(x,0) = g(x)\,. \end{gathered}$$

(a) Assume a solution in the form $u(x,t) = X(x)T(t)$ and find the differential equations that X and T must satisfy. We plug into the equation and divide by XT to get

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\sigma^2\,.$$

We choose a negative constant so the solution decays in time. The boundary and initial conditions give us

$$\begin{aligned} X'(0) &= -X(0)\,, \\ X'(1) &= 2X(1)\,, \\ T(0)X(x) &= g(x)\,. \end{aligned}$$

(b) Find the relationship the eigenvalues must satisfy. The equation for X is

$$X'' + \sigma^2 X = 0$$

so $X(x) = A\cos(\sigma x) + B\sin(\sigma x)$. We apply the boundary conditions to get

$$\begin{gathered} B\sigma = -A\,, \\ -A\sigma\sin(\sigma) + B\sigma\cos(\sigma) = 2A\cos(\sigma) + 2B\sin(sigma)\,. \end{gathered}$$

Or in matrix notation

$$\begin{pmatrix} 1 & -\sigma \\ -\sigma\sin(\sigma) - 2\cos(\sigma) & \sigma\cos(\sigma) - 2\sin(\sigma) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For there to be nontrivial solutions, we require the determinant to be 0 or

$$\sigma\cos(\sigma) - 2\sin(\sigma) - \sigma^2\sin(\sigma) - 2\sigma\cos(\sigma) = 0$$

Here the eigenvalues λ are the square root of σ.

4. The displacement $y(x,t)$ of a vibrating string under the influence of gravity must satisfy

$$y_{tt} = y_{xx} - g\,.$$

If the endpoints are fixed then $y(0,t) = y(L,t) = 0$.

(a) If the string is stationary ($y_{tt} = 0$), find the displacement of the string, $\phi(x)$.

ϕ satisfies the equation,

$$\phi''(x) = g\,, \phi(0) \qquad\qquad = \phi(L) = 0\,.$$

So $\phi(x) = \frac{1}{2}gx^2 - \frac{Lg}{2}x$.

(b) If the string is released with no initial velocity or displacement then $y(x,0) = y_t(x,0) = 0$. Let $v(x,t) = y(x,t) - \phi(x)$. What equation must v satisfy?

First $v_{tt} = y_{tt}$ and $v_{xx} = y_{xx} - g$, so the equation for v is

$$\begin{aligned} v_{tt} &= v_{xx}\,, \\ v(0,t) &= y(0,t) - \phi(0) = 0\,, \\ v(L,t) &= y(L,t) - \phi(L) = 0\,, \\ v(x,0) &= y(x,0) - \phi(x) = -\phi(x)\,, \\ v_t(x,0) &= y_t(x,0) - 0 = 0\,. \end{aligned}$$

(c) Solve for $v(x,t)$ and thus $y(x,t)$.

We assume $v(x,t) = X(x)T(t)$ and we get

$$\begin{aligned} X''(x) + \sigma^2 X(x) &= 0\,, \\ T''(t) + \sigma^2 T(t) &= 0\,. \end{aligned}$$

The constant σ^2 must be positive for non-trivial solutions. Applying the boundary conditions gives us $\sigma_n = n\pi/L$ and $X_n(x) = \sin(n\pi x/L)$ and $T_n = A_n \cos(n\pi x/L)$. We can then write the solution to the original problem as

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)\sin(n\pi x/L) + \phi(x)\,,$$

where,

$$A_n = \frac{2}{L}\int_0^L (-\phi(x))\sin(n\pi x/L)\,dx\,.$$

5. Consider the eigenvalue problem

$$\begin{aligned} u'' + u' + u &= -\lambda u\,, \quad 0 < x < 1\,, \\ u(0) &= 0\,, \\ u(1) + 2u(1) &= 0\,. \end{aligned}$$

(a) Recast the problem in the form as a Sturm-Liouville problem.

To write the problem in Sturm-Liouville form, we multiply the equation by $e^{\int 1\,dx} = e^x$ to get

$$e^x u'' + e^x u' + e^x u = -\lambda e^x u\,,$$
$$(e^x u')' + e^x u = -\lambda e^x u\,.$$

so $p(x) = e^x$, $q(x) = -e^x$ and $r(x) = e^x$.

(b) Find the eigenvalues, λ_i and eigenfunctions $\phi_i(x)$.

We can solve $u'' + u' + (1+\lambda)u = 0$ by subbing in $u = e^{rx}$. We get $r^2 + r + (1+\lambda) = 0$ or

$$r_{1,2} = \frac{-1 \pm \sqrt{3+4\lambda}i}{2}$$

The general solution is then

$$u = e^{-x/2}(A\cos(\sqrt{(}\sqrt{3+4\lambda}/2x) + B\sin(\sqrt{3+4\lambda}/2))$$

Now we apply the boundary conditions. There is a typo and the problem is much simpler if we take $3u(1) = 0$, so I will work with $u(1) + 2u'(1) = 0$ as I originally intended. $u(0) = 0$ implies $u = Be^{-x/2}\sin(\omega x)$, where $\omega = \sqrt{3+4\lambda}/2$. We now apply the other boundary condition to get

$$B(e^{-1/2}\sin(\omega) + 2(-\frac{1}{2}e^{-x/2}\sin(\omega x) + e^{-x/2}\cos(\omega x)) = 0$$

or $\omega_n = \frac{(2n-1)\pi}{2}$, $n = 1, 2, \ldots$. The eigenpairs are then given by

$$\lambda_n = \left(\left(\frac{(2n-1)\pi}{2}\right)^2 - 3\right)/4\,,$$
$$\phi_n(x) = e^{-x/2}\sin\left(\frac{(2n-1)\pi}{2}x\right)\,.$$

(c) Given a reasonable function $f(x)$ express it as an eigenfunction expansion $f = \sum_{n=0}^{\infty} c_i\phi_i(x)$. The formula for c_i should be in terms of f and know quantities.

The eigenfunctions are orthogonal under the dot product defined by $< u, v >= \int_0^1 u(x)v(x)e^x\,dx$, so if we wish to write $f(x) = \sum_{n=1}^{\infty} f_n\phi_n(x)$, then $f_n = \frac{<f,\phi_n>}{<\phi_n,\phi_n>}$ where $< u, v >$ and $\phi_n(x)$ are defined above.

6. Denote the population of a desirable species by $x(t)$ and an undesirable competing species by $y(t)$. If resources are used to reduce the population of y, then we can model the dynamics of the populations by

$$x' = r(1 - \alpha x - \beta y)x\,,$$
$$y' = r(1 - \alpha y - \beta x)y - \mu y\,.$$

Assume $\alpha > \beta > 0$ and r and μ are both greater than 0.

(a) Determine the four equilibria of the system.

It is clear that $(x, y) = (0, 0)$ is one solution. If we set $x = 0$, we get another one at $(x, y) = (0, \frac{1}{\alpha}\left(\frac{r-\mu}{r}\right)$. If we set $y = 0$, we get $(x, y) = \left(\frac{1}{\alpha}, 0\right)$. The final equilibrium is given by

$$(x, y) = \left(\frac{1}{\alpha} - \frac{\beta}{\alpha}\frac{\alpha\left(1 - \frac{\mu}{r}\right) - \beta}{\alpha^2 - \beta^2}, \frac{\alpha\left(1 - \frac{\mu}{r}\right) - \beta}{\alpha^2 - \beta^2}\right)$$

(b) Show that if μ is large enough then there are only two viable equilibria (in the first quadrant since population can't be negative).

Since $\alpha > \beta > o$, if $\mu > r$ the only equilibria with two positive populations will be $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (\frac{1}{\alpha}, 0)$.

(c) Assume μ is large enough so that there are only two physically viable equilibria as above. Compute the linearized stability of both.

We compute the Jacobean as

$$\begin{pmatrix} r - 2\alpha r x - \beta y & -r\beta x \\ -r\beta y & r - 2\alpha r y - \beta x - \mu \end{pmatrix}$$

Look at $(x, y) = (0, 0)$,

$$J(0, 0) = \begin{pmatrix} r & 0 \\ 0 & r - \mu \end{pmatrix}$$

So we have one eigenvalue of r which is positive and one eigenvalue of $r - \mu$ which is negative. So this point is a saddle and is unstable.

Look at $(x, y) = (\frac{1}{\alpha}, 0)$

$$J(\frac{1}{\alpha}, 0) = \begin{pmatrix} -r & -\frac{r\beta}{\alpha} \\ 0 & r - \mu - \frac{\beta}{\alpha} \end{pmatrix}$$

The two eigenvalues are $-r$ which is negative and $r - \mu - \frac{\beta}{\alpha}$ which is also negative, so the origin is stable. So if we cull the undesirable species to much the only stable equilibrium is the origin.

7. The 2nd order midpoint Runge-Kutta method for approximating $y' = f(y, t)$ is given by

$$y_{n+1} = y_n + hf(y_n + \frac{h}{2}f(y_n, t_n), t_n + \frac{h}{2})$$

where h is the step size, $t_n = nh$ and y_0 is given by the initial condition.

(a) Apply the method to $y' = \lambda y$. Give the equation for y_{n+1} in terms of y_n.

Here $f(y, t) = \lambda y$, so we get

$$y_{n+1} = y_n + h(\lambda(y_n + \frac{h}{2}(\lambda y_n))) ,$$

$$y_{n+1} = y_n(1 + h\lambda + \frac{(h\lambda)^2}{2}) .$$

(b) What must $h\lambda$ satisfy to ensure $\left|\frac{y_{n+1}}{y_n}\right| < 1$.

From the above we will just require

$$|1 + h\lambda + \frac{(h\lambda)^2}{2}| < 1$$

Remember that $h\lambda$ is in general a complex number and $|a + ib| = \sqrt{a^2 + b^2}$.

In addition to questions like the above the test will also have a page of short answer type questions.

A formula sheet is allowed.

I will be posting additional questions soon.