**Math 3120 Practise Test**

1. Find two independent power series solutions for,

$$y'' + xy = 0\,,$$

and discuss convergence (give the first 4 nonzero terms, you don't need to find a general formula for the $n^{th}$ term).

First, $x = 0$ is a regular point, so we have a solution in the form $y = \sum_{n=0}^{\infty} a_n x^n$. We substitute into the equation to get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0\,,$$
$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=3}^{\infty} a_{n-3} x^{n-2} = 0\,,$$
$$\sum_{n=3}^{\infty} (a_n n(n-1) + a_{n-3})x^{n-2} + 2a_2 = 0\,.$$

So we have $a_2 = 0$ and

$$a_n = -\frac{a_{n-3}}{n(n-1)}\,.$$

Two get 2 independent solutions, we can first set $a_0 = 1$ and $a_1 = 0$.

$$a_3 = -\frac{a_0}{3\cdot 2} = -\frac{1}{6}\,,$$
$$a_6 = -\frac{a_3}{6\cdot 5} = \frac{1}{180}\,,$$
$$a_9 = -\frac{a_6}{9\cdot 8} = -\frac{1}{72\cdot 180}\,,$$

For the second solution, we take $a_0 = 0$ and $a_1 = 1$ then we have

$$a_4 = -\frac{a_1}{4\cdot 3} = -\frac{1}{12}\,,$$
$$a_7 = -\frac{a_4}{7\cdot 6} = \frac{1}{7\cdot 72}\,,$$
$$a_{10} = -\frac{a_7}{10\cdot 9} = -\frac{1}{90\cdot 7\cdot 72}\,.$$

and the two independent solutions can be written as

$$y = c_1\left(1 - \frac{x^3}{3!} + \frac{4x^6}{6!} - \frac{7\cdot 4x^9}{9!} + \cdots\right) + c_2\left(x - \frac{2x^4}{4!} + \frac{2\cdot 5x^7}{7!} - \frac{2\cdot 5\cdot 8x^{10}}{10!} + \cdots\right)$$

Since the point is regular, the solutions will converge for all $x$.

2. For the differential equation,

$$2xy'' - (1+x)y' + 2y = 0\,,$$

(a) Show $x = 0$ is a regular singular point.

Here $p(x) = -\frac{1+x}{2x}$ and $q(x) = \frac{1}{x}$, both are singular at $x = 0$, so $x = 0$ is a singular point. However,

$$p_0 = \lim_{x\to 0} xp(x) = -\frac{1}{2} < \infty\,,$$
$$q_0 = \lim_{x\to 0} x^2 q(x) = 0 < \infty\,.$$

So $x = 0$ is a regular singular point.

(b) Determine the indicial equation and its two roots.
The indicial equation is given by

$$\begin{aligned} r(r-1)+p_0r+q_0 &= 0\,,\\ r(r-1)-\frac{1}{2}r &= 0\,,\\ r(r-\frac{3}{2}) &= 0\,, \end{aligned}$$

So the two roots are $r=0$ and $r=\frac{3}{2}$.

(c) Write out the form of the series solution. If possible give the form for 2 independent solutions.
Since there are two distinct roots which don't differ by an integer, the method of Frobenious gives two independent solutions. The two forms of the solution will be

$$\begin{aligned} y_1 &= \sum_{n=0}^{\infty} a_n x^n\,,\\ y_2 &= \sum_{n=0}^{\infty} b_n x^{n+3/2}\,. \end{aligned}$$

(d) Find the recurrence relation.
I'll find both relations at once, by subbing in $y=\sum_{n=0}^{\infty} a_n x^{n+r}$ then taking both valid values for $r$. We sub in to get

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r-1}-\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}-\sum_{n=0}^{\infty} a_n(n+r)x^{n+r}+\sum_{n=0}^{\infty} 2a_n x^{n+r}\,,$$

$$\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r-1}-\sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}-\sum_{n=1}^{\infty} a_{n-1}(n+r-1)x^{n+r-1}+\sum_{n=1}^{\infty} 2a_{n-1}x^{n+r-1}\,,$$

So the recurrence relations are given by

$$a_n=\frac{r+n-3}{(2r+2n-3)(r+n)}a_{n-1}\,,$$

where $r=0$ or $r=3/2$.

3. Given the following heat transfer problem:

$$\begin{aligned} u_t &= (0.5)u_{xx}, \quad 0<x<4\,,\\ u(0,t) &= 0\,,\\ u(4,t) &= 100\,,\\ u(x,0) &= 25x+80\sin(\pi x)\cos(\pi x)\,. \end{aligned}$$

(a) Construct a linear function $v(x)$ such that $u(x,t)-v(x)$ is 0 at $x=0$ and $x=4$.
We want the function $v$ to be 0 at $x=0$ and 100 at $x=4$, so we set $v(x)=25x$.

(b) Determine the equation the function $w(x,t)=u(x,t)-v(x)$ must satisfy.
First we note that since $v(x)$ is a linear function of $x$, $w_t=u_t$ and $w_{xx}=u_{xx}$. We just need to work out the boundary conditions and the initial condition that $w$ must satisfy. $w(0,t)=u(0,t)-v(0)=0$ and $w(4,t)=u(4,t)-v(4)=0$. Finally $w(x,0)=u(x,0)-v(x)=80\sin(pix)\cos(\pi x)$. So $w$ must satisfy

$$\begin{aligned} w_t &= (0.5)w_{xx}\,, \quad 0<x<4\,,\\ w(0,t) &= 0\,,\\ w(4,t) &= 0\,,\\ w(x,0) &= 80\sin(\pi x)\cos(\pi x)\,. \end{aligned}$$

(c) Use separation of variable and Fourier series to find a solution for $w$ and then find $u$.
We have already solved this equation, it is just a sin series solution.

$$w = \sum_{n=1}^{\infty} a_n e^{-0.5\frac{n^2\pi^2}{16}t} \sin\left(\frac{n\pi}{4}x\right) ,$$

where

$$a_n = \frac{2}{4}\int_0^4 80\sin(\pi x)\cos(\pi x)\sin\left(\frac{n\pi}{4}x\right) ,$$

Then $u(x,t) = w(x,t) + 25x$

4. Consider the eigenvalue problem,

$$\begin{aligned} u'' + 2u' + u &= -\lambda u\,, \\ u(0) &= 0\,, \\ u(1) &= 0\,. \end{aligned}$$

(a) Recast the differential equation in the form $L(u) = (p(x)u')' - q(x)u = -\lambda r(x)u$. What are $p$, $q$ and $r$?
We need to multiply the equation by $e^{\int 2\,dx} = e^{2x}$.

$$\begin{aligned} e^{2x}u'' + 2e^{2x}u' + e^{2x}u &= -\lambda e^{2x}u\,, \\ (e^{2x}u')' + e^{2x}u &= -\lambda e^{2x}u\,. \end{aligned}$$

So $p(x) = e^{2x}$, $q(x) = -e^{2x}$ and $r(x) = e^{2x}$.

(b) Determine the eigenpairs.
To find the eigenpairs, we try a solution in the form $y = e^{rx}$. This will result in the equation

$$r^2 + 2r + (1+\lambda) = 0$$

The solutions are

$$r_{1,2} = \frac{-2 \pm \sqrt{4 - 4(1+\lambda)}}{2}$$

or

$$r_{1,2} = -1 \pm \sqrt{-\lambda}\,.$$

Now if $\lambda \leq 0$ the solutions must be identically zero. So we will require $\lambda > 0$. After applying the boundary conditions, we will get the eigenpairs are

$$\lambda_n = n^2\pi^2\,, \quad u_n = e^{-x}\sin(n\pi x)$$

(c) Verify that the eigenfunctions are orthogonal with respect to the appropriate dot product.
The appropriate dot product for this problem is $< f, g >= \int_0^1 fge^{2x}\,dx$ so

$$\begin{aligned} < u_n, u_m > &= \int_0^1 (e^{-x}\sin(n\pi x))(e^{-x}\sin(m\pi x))e^{2x}\,dx\,, \\ &= \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx\,, \\ &= \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \end{cases} \end{aligned}$$

So the eigenfunctions are orthogonal.

In addition to 4 questions like the above the test will also have a page of short answer type questions.
A formula sheet is allowed.