

## 1. Short Answer Questions

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- (a) Consider the following homogeneous partial differential equation with inhomogeneous boundary conditions:

$$\begin{aligned} u_t &= u_{xx}, & 0 \leq x \leq 1, & \quad t > 0, \\ u(0, t) &= 1, & u(1, t) &= 5, & \quad u(x, 0) = 1. \end{aligned}$$

- i. Find a function  $\phi(x, t)$  such that  $v(x, t) = u(x, t) - \phi(x)$  will satisfy homogeneous boundary conditions.

$$\phi(x) = 1 + 4x$$

- ii. The function  $v$  will then satisfy the partial differential equation with initial conditions given by:

$$v(x, 0) = u(x, 0) - \phi(x) = 1 - (1 + 4x) = -4x$$

- (b) Consider the following Sturm-Liouville eigenvalue problem,

$$\begin{aligned} (x^3 \phi'(x))' + x \phi(x) &= -x^2 \lambda \phi(x), \\ \phi(1) &= \phi(2) = 0. \end{aligned}$$

The eigenfunctions of this problem are orthogonal under the dot product

$$\langle u, v \rangle = \int_1^2 uvx^2 dx$$

- (c) Consider the function

$$y(x) = \begin{cases} 0 & x < 1, \\ 5 & x \geq 1, \end{cases}$$

The Fourier series of  $y(x)$  will converge to what at  $x = 1$ .

(c) \_\_\_\_\_

$[\frac{5}{2}]$

- (d) The series solution about  $x = 1$  for the differential equation

$$y'' + \frac{1}{1+4x^2}y' + \frac{x}{4+x}$$

has what radius of convergence?

(d) \_\_\_\_\_

$[\frac{\sqrt{5}}{2}]$

- 10 2. Find the first two non-zero terms in a series solution, centred at  $x = 0$ , for the differential equation

$$(x + 2) \frac{d^2 y}{dx^2} - y = 0,$$

$$y(0) = 0,$$

$$y'(0) = 3.$$

Discuss the radius of convergence.

**Solution:** The point  $x = 0$  is a regular point, so we may set,

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$$

Substituting in

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-1} + \sum_{n=2}^{\infty} 2a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n,$$

$$\sum_{n=1}^{\infty} a_{n+1} n(n+1) x^n + \sum_{n=0}^{\infty} 2a_{n+2} (n+1)(n+2) x^n - \sum_{n=0}^{\infty} a_n x^n,$$

$$\sum_{n=1}^{\infty} (a_{n+1} n(n+1) + 2a_{n+2} (n+1)(n+2) - a_n) x^n + 2a_2(2) - a_0.$$

From the initial conditions and the expansion, we have

$$a_0 = 0,$$

$$a_1 = 3,$$

$$a_2 = 0,$$

$$a_{n+2} = \frac{a_n - a_{n+1} n(n+1)}{2(n+1)(n+2)}$$

Using the recurrence relation we have  $a_3 = \frac{1}{4}$  and the first two non zero terms of the expansion are given by

$$y = 3x + \frac{x^3}{4} + \dots$$

and since the only singular point is at  $x = -2$ , the series must converge in  $-2 < x < 2$ .

- 10 3. Consider the differential equation

$$4xy'' + 2y' + y = 0.$$

- (a) Show  $x = 0$  is a regular singular point.  
 (b) Find the roots of the indicial equation.  
 (c) Give the forms of the series expansion for two independent solutions.

**Solution:**

- (a) Here  $p(x) = \frac{1}{2x}$  and  $q(x) = \frac{1}{4x}$ , since  $\lim_{x \rightarrow 0} p(x) = \lim_{x \rightarrow 0} q(x) = \infty$ ,  $x = 0$  is a singular point. Since  $p_0 = \lim_{x \rightarrow 0} xp(x) = \frac{1}{2}$  and  $q_0 = \lim_{x \rightarrow 0} x^2q(x) = 0$ ,  $x = 0$  is a regular singular point.

- (b) To find the roots of the indicial equation, we solve

$$\begin{aligned} r(r-1) + p_0r + q_0 &= 0, \\ r^2 - r + \frac{r}{2} &= 0, \\ r^2 - \frac{r}{2} &= 0, \\ r = 0, \text{ or } r &= \frac{1}{2} \end{aligned}$$

- (c) Since the two roots are distinct and don't differ by an integer, the forms of the solutions are

$$y_1 = \sum_{n=0}^{\infty} a_n x^n \text{ and } y_2 = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}.$$

- 10 4. The equation of motion for a damped vibrating string is given by

$$u_{tt} + \beta u_t = c^2 u_{xx}$$

where  $\beta$  is a damping constant.

- (a) Use separation of variables to turn this partial differential equations into two ordinary differential equations.  
 (b) If the sting is fixed at  $x = 0$  and  $x = 1$  (i.e.  $u(0, t) = u(1, t) = 0$ ), find the eigenvalues and eigenfunctions for the problem found in (a) .

**Solution:**

(a) We let  $u(x, t) = X(x)T(t)$  and plug into the equation,

$$\begin{aligned} X(x)T''(t) + \beta X(x)T'(t) &= c^2 X''(x)T(t), \\ \frac{X(x)T''(t) + \beta X(x)T'(t)}{X(x)T(t)} &= \frac{c^2 X''(x)T(t)}{X(x)T(t)}, \\ \frac{T''(t) + \beta T'(t)}{c^2 T(t)} &= \frac{X''(x)}{X(x)} = -\sigma, \end{aligned}$$

(b) First we note that we will require  $\sigma > 0$  for a nontrivial solution. If  $\sigma < 0$ , we set  $\mu = \sqrt{-\sigma}$  and the solution for  $X$  is

$$X(x) = Ae^{\mu x} + Be^{-\mu x}$$

Applying the first boundary condition implies  $A = -B$  then the second gives

$$A(e^{\mu} - e^{-\mu}) = 0$$

This implies  $\mu = 0$  and  $X(x)$  is then a constant. The boundary conditions imply this constant is 0. If  $\sigma = 0$ , then the solution is a straight line. The only line going through  $(0, 0)$  and  $(1, 0)$  is again identically 0. So the solution must be given by

$$X(x) = A \cos(\sqrt{\sigma}x) + B \sin(\sqrt{\sigma}x)$$

Applying the first boundary condition gives  $A = 0$  and the second implies  $\sqrt{\sigma} = n\pi$ ,  $n = 1, 2, \dots$ . The eigenpairs are then given by  $\{n^2\pi^2, \sin(n\pi x)\}_{n=1}^{\infty}$

I will just add a little note for interest. Given the eigenvalues, the equation for  $T(t)$  is

$$T''(t) + \beta T'(t) + (n\pi c)^2 T = 0$$

If  $m$  is the integer such that  $4m\pi c < \beta < 4(m+1)\pi c$ , then the general solution to the equation is given by.

$$\begin{aligned} &\sum_{n=1}^m \left( a_n e^{\frac{-\beta + \sqrt{\beta^2 - 4(n\pi c)^2}}{2} t} + b_n e^{\frac{-\beta - \sqrt{\beta^2 - 4(n\pi c)^2}}{2} t} \right) \sin(n\pi x) \\ &+ \sum_{n=m+1}^{\infty} e^{-\beta t/2} \left( a_n \cos\left(\sqrt{4(n\pi c)^2 - \beta^2}/2t\right) + b_n \sin\left(\sqrt{4(n\pi c)^2 - \beta^2}/2t\right) \right) \sin(n\pi x) \end{aligned}$$

The first  $m$  modes decay immediately. The remaining modes decay in an oscillating fashion.

- 10 5. Consider the non-Sturm Liouville eigenvalue problem

$$x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} + \lambda \phi = 0, \quad \phi(1) = \phi(2) = 0, \quad (1)$$

- (a) Write (1) in standard Sturm-Liouville form.  
 (b) Equation (1) is an Euler type equation and has solutions of the form  $\phi = x^r$  for some values of  $r$ . Find the eigenvalues and eigenfunctions of (1).  
 (c) For any given a function  $f(x)$  which is continuous on  $[1, 2]$ , write the eigenfunction expansion for  $f$  (be sure and include all the formulae for any constants in the expansion).

**Solution:**

- (a) We first multiply by  $\frac{1}{x^2}$  to get the equation in the form

$$\phi'' + \frac{1}{x} \phi' = -\frac{1}{x^2} \lambda \phi.$$

To put this equation into standard Sturm-Liouville form, we multiply by  $e^{\int \frac{dx}{x}} = x$ . We then have,

$$\begin{aligned} x\phi'' + \phi' &= -\lambda \frac{1}{x} \phi, \\ (x\phi')' &= -\lambda \frac{1}{x} \phi. \end{aligned}$$

The important thing to note at this stage is the weight function  $r(x) = \frac{1}{x}$ .

- (b) If we plug in  $y = x^r$  as a solution, we get the following equation

$$r(r-1) + r + \lambda = 0$$

Which means  $r = \pm\sqrt{-\lambda}$ . We can consider the case  $\lambda = -\mu < 0$  which means the general solution is of the form

$$y = Ae^{\mu x} + Be^{-\mu x}.$$

$\phi(1) = 0$  implies that  $A = -B$  and  $\phi(2) = 0$  implies that  $A(2^\mu - 2^{-\mu}) = 0$  which is only possible if  $\mu = 0$ . This would make the solution a constant which must be zero to satisfy the boundary condition. If we consider  $\lambda = 0$  again we get a constant solution which must be 0. The only possible solution is  $\lambda > 0$  which means:

$$\begin{aligned} y &= Ax^{i\sqrt{\lambda}}, \\ &= Ae^{i\sqrt{\lambda}\ln(x)}, \\ &= A(\cos(\sqrt{\lambda}\ln(x)) + i\sin(\sqrt{\lambda}\ln(x))). \end{aligned}$$

So we may construct a general solution of the form

$$y = A \cos(\sqrt{\lambda} \ln(x)) + B \sin(\sqrt{\lambda} \ln(x))$$

The condition  $y(1) = 0$  implies  $A = 0$  and the condition  $y(2) = 0$  implies  $\sqrt{\lambda} \ln(2) = n\pi$  where  $n = 1, 2, \dots$ . So the eigenpairs are given by

$$\left\{ \frac{n^2\pi^2}{\ln(2)}, \sin\left(\frac{n\pi}{\ln(2)} \ln(x)\right) \right\}_{n=1}^{\infty}$$

(c) Given any reasonable function  $f$ , we may write

$$f = \sum_{n=1}^{\infty} a_n \phi_n(x),$$

Where  $\phi_n(x)$  are the eigenfunctions given above. The coefficients  $a_n$  are given by

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\int_1^2 f(x) \phi_n(x) \frac{1}{x} dx}{\int_1^2 (\phi_n(x))^2 \frac{1}{x} dx}.$$