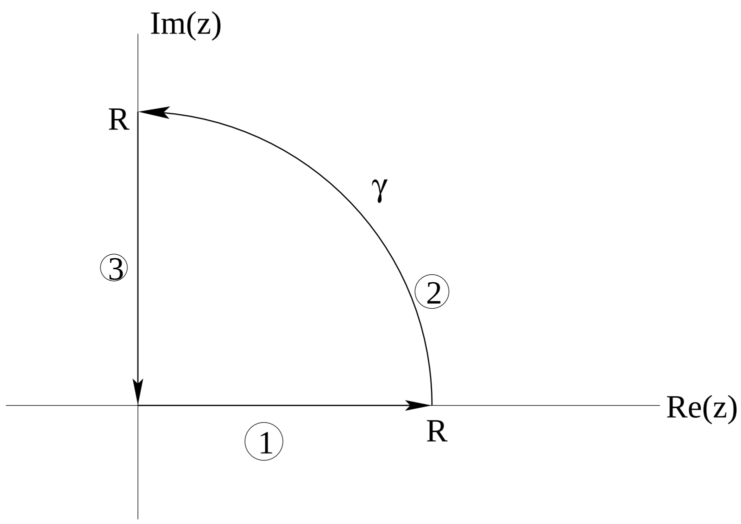
The Argument Principle and Roots of Polynomials

Theorem 1 *Argument Principle Let the function $w(z)$ be analytic in a simply connected domain G. Let $w(z)$ have at most a finite number of zeros in G. If γ is an oriented path in G not passing through any of the zeros of $w(z)$, the increment of* $\arg(w(z))$*, as z describes the curve γ, is given by,*

$$\Delta_\gamma(\arg(w)) = 2\pi N\,, \tag{1}$$

where N is the number of zeros of $w(z)$ inside of γ counting multiplicity.

We can use this result to find the number of zeros a polynomial has in a given region of the complex plane. For example if we wish to find the number of zeros of $f(z) = z^4+z+1$ in the first quadrant of the complex plane, we would consider the path, γ,



and let $R \to \infty$. We consider the three parts of the path separately.

- Along 1, $f(z)$ is real, so the argument is 0. There is no change in the argument.
- Along 2, $z = Re^{i\theta}$ for $0 \le \theta \le \frac{\pi}{2}$. So,

$$\begin{aligned} f(z) &= R^4e^{4i\theta} + Re^{i\theta} + 1\,, \\ &= R^4e^{4i\theta}\left(1 + \frac{1}{R^3e^{3i\theta}} + + \frac{1}{R^4e^{4i\theta}}\right)\,. \end{aligned}$$

  So as $R \to \infty$, $f(z) = R^4e^{4i\theta}$ along 2. So as θ goes from 0 to $\frac{\pi}{2}$, $\arg(f(z))$ goes from 0 to 2π.

- Along 3, $z = iy$, so $f(z) = (y^4 + 1) + iy$. We note that along 3, the real part of $f(z)$ and the complex part of $f(z)$ remain positive, so we stay in the first quadrant. We also note that along 3,

$$\arg(f(iy)) = \arctan\left(\frac{y}{y^4 + 1}\right),$$

with the correct choice of branch for arctan.

Now we when we start 3, $\arg(f(z)) = \arctan\left(\frac{R}{R^4+1}\right)$. As $R \to \infty$ we get the $\arg(f(z)) \to 0$. We will use 2π to remain continuous with the argument at the end of 2. As finish the last part of the path we again get the $\arg(f(z)) \to 0$. Since we must remain in the first quadrant for the entire time we are on 3, we have the argument of $f(z)$ doesn't change on this part of the path. So the total change of argument of $f(z)$ along γ is 2π, so We only have one zero in the first quadrant.