**Math 4200/5200 -Differential Equations-Qualitative Theory**

Homework #3 Due Friday March 5

1. Consider the system
$$\dot{x} = Ax + f(x)\,, \quad x \in \mathbb{R}^2\,,$$
where
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$f$ is $C^3$ and $f(x) = O(|x|^2)$. Find a near identity transformation $x = h(y) = y + h_2(y)$, where the components of $h_2$ are homogeneous polynomials of degree two in $y = (y_1, y_2)$, that transforms the system into Poincare normal form up to second order
$$\dot{y_1} = y_2 + O(|y_1, y_2|^3)\,,$$
$$\dot{y_2} = ay_1^2 + by_1y_2 + O(|y_1, y_2|^3)\,.$$
Find the Poincare normal form coefficients $a$ and $b$ in terms of partial derivatives of $f$ at $x = 0$. Sketch phase portraits of the *truncated* transformed system for various $a$ and $b$ assuming they are both nonzero.

2. In this problem we consider a model (the Brusellator) for a chemical reaction. If the container is well stirred, diffusion of chemicals can be ignored, and the equation takes the form,
$$\dot{x} = a - (b+1)x + x^2y\,,$$
$$\dot{y} = bx - x^2y\,,$$
where $a$, $b$, $x$ and $y$ represent concentrations if chemicals. $a$ and $b$ are assumed to remain constant, so are considered to be parameters. Here we fix $a > 0$ and treat $b$ as a (positive) bifurcation parameter.
   (a) Find the unique equilibrium, linearize about the equilibrium and show that the linearization has pure imaginary eigenvalues when $b = 1 + a^2$.
   (b) Analyze the Hopf bufurcation that occurs at $b = 1 + a^2$. Draw a branching diagram, and sketch local phase portraits (up to topological equivalence) fo the Brusellator for $b < 1 + a^2$, $b = 1 + a^2$ and $b > 1 + a^2$, $b$ near $1 + a^2$.

3. Consider the one-parameter family of vector fields:
$$\dot{x_1} = 2x_1 + (2+\alpha)x_2\,,$$
$$\dot{x_2} = x_1 + x_2 + x_1^4\,.$$
where $\alpha$ is a parameter. Use centre manifold theory to study the dynamics near the origin, for $\alpha$ near 0 ($< 0, = 0, > 0$). Determine the dynamics on the centre manifold, using the implicit function theorem to justify the neglect of higher order terms. Draw two-dimensional phase portraits for $\alpha$ near 0, and draw a branching diagram.