Math 4220/5220 -Introduction to PDE's

Homework #1 Solutions

1. Find the most general solution to the following PDEs:

(a) $au_x + bu_y + cu = 0$ where a, b and c are constants.
For this problem, we let $\eta = y - \frac{b}{a}x$ and $\xi = x$. Then

$$u_x = u_\eta(-\frac{b}{a}) + u_\xi \,,$$
$$u_y = u_\eta \,.$$

The equation in the new variables is then given by

$$au_\xi + cu = 0$$

The solution is given by

$$u = f(\eta)e^{-\frac{c}{a}\xi}$$

or

$$u = f(y - \frac{b}{a}x)e^{-\frac{c}{a}x}$$

(b) $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.
We let $\eta = y - x$ and $\xi = x$. So $y = \eta + \xi$. Under the new variables the equation is given by

$$u_\xi + u = e^{2\eta+3\xi} \,.$$

Thus

$$\begin{aligned}\left(e^\xi u\right)_\xi &= e^{2\eta+4\xi} \,,\\ e^\xi u &= \frac{1}{4}e^{2\eta+4\xi} + f(\eta) \,,\\ u &= \frac{1}{4}e^{2\eta+3\xi} + e^{-\xi}f(\eta) \,,\\ u &= \frac{1}{4}e^{2y+x} + e^{-x}f(y - x) \,.\end{aligned}$$

Now we use the initial data to solve for f.

$$\begin{aligned}u(x, 0) &= \frac{1}{4}e^x + e^{-x}f(-x) = 0 \,,\\ f(-x) &= -\frac{1}{4}e^{2x} \,,\\ f(x) &= -\frac{1}{4}e^{-2x} \,.\end{aligned}$$

The solutions is then given by

$$u = \frac{1}{4}e^{2y+x} - e^{-x}\frac{1}{4}e^{-2(y-x)}$$

(c) $u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2$.
We let $\eta = y - 2x$ and $\xi = x$. So $y = \eta + 2\xi$. Then we have

$$\begin{aligned} u_\xi - \eta u &= 5\xi\eta - 2\eta^2 \,, \\ \left(e^{\xi\eta} u\right)_\xi &= (5\xi\eta - 2\eta^2)e^{\xi\eta} \,, \\ u &= 5\xi - 2\eta - \frac{5}{\eta} + e^{\xi\eta} f(\eta) \,, \\ u &= 9x - 2y - \frac{5}{y - 2x} + f(y - 2x)e^{xy - 2x^2} \,. \end{aligned}$$

2. Consider the equation $3u_y + u_{xy} = 0$.
   (a) What is its type?
   Since the second order terms are already in canonical form, we can readily tell this is an elliptic equation.
   (b) Find the general solution. (Hint: Substitute $v = u_y$.)
   With the suggested substitution, we have the equation

$$\begin{aligned} 3v + v_x &= 0 \,, \\ (e^{3x} v)_x &= 0 \,, \\ v &= f(y)e^{-3x} \,. \end{aligned}$$

   Then

$$\begin{aligned} u_y &= f(y)e^{-3x} \,, \\ u &= \int_0^y f(\eta)e^{-3x} \, d\eta + u(x, 0) \,, \end{aligned}$$

   (c) With the auxiliary conditions $u(x, 0) = e^{-3x}$ and $u_y(x, 0) = 0$, does a solution exist? Is it unique?
   Now $u = \int_0^y f(\eta)e^{-3x} \, d\eta + e^{-3x}$, so

$$\begin{aligned} u_y(x, 0) &= e^{-3x}(f(0) + 1) = 0 \,, \\ f(0) &= -1 \,. \end{aligned}$$

   So any function with $f(0) = -1$ will satisfy the conditions. The equations does have solutions, but they are not unique.

3. The PDE for a 3-dimensional radially symmetric wave is given by,

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right) \,.$$

   (a) Solve $u_{tt} = c^2 u_{xx}$, $u(x, 0) = e^x$, $u_t(x, 0) = \sin(x)$.
   We have,

$$\begin{aligned} u &= f(x + ct) + g(x - ct) \,, \\ u(x, 0) &= f(x) + g(x) = e^x \,, \\ u_t(x, 0) &= cf'(x) - cg'(x) = \sin(x) \,. \end{aligned}$$

So $f'(x) = \frac{1}{2}e^x + \frac{\sin(x)}{2c}$. Then $f(x) = \frac{1}{2}e^x - \frac{\cos(x)}{2c} + C$ and $g(x) = \frac{1}{2}e^x + \frac{\cos(x)}{2c} - C$. The solutions is then:

$$u(x,t) = \frac{1}{2}(e^{x+ct} + e^{x-ct}) + \frac{1}{2c}\left(\cos(x-ct) - \cos(x+ct)\right)$$

(b) Change variables $v = ru$ the get the equation for v: $v_{tt} = c^2 v_{rr}$
We solve

$$\begin{aligned} v_{tt} &= ru_{tt}\,, \\ v_r &= ru_r + u\,, \\ v_{rr} &= ru_{rr} + 2u_r\,. \end{aligned}$$

Hence,

$$\begin{aligned} v_{tt} - c^2 v_{rr} &= ru_{tt} - c^2(ru_{rr} + 2u_r)\,, \\ &= r(u_t t - c^2(u_{rr} + \frac{2}{r}u_r)\,, \\ &= 0\,. \end{aligned}$$

(c) Use this change of variables to solve the spherically symmetric wave equation in 3-dimensions with the initial conditions $u(r,0) = \phi(r)$, $u_t(r,0) = \psi(r)$ where ϕ and ψ are even functions of r.
We have $v = ru$ satisfies:

$$\begin{aligned} v_{tt} &= c^2 v_{rr}\,, \\ v(r,0) &= r\phi(r)\,, \\ v_r(r,0) &= r\psi(r)\,. \end{aligned}$$

Since ϕ and ψ are even functions of r, the solution will satisfy $u_r(0,t) = 0$ and is given by,

$$u = r^2\left(\frac{1}{2}(\phi(x+ct) + \phi(x-ct)\right) + \frac{r}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

4. For each of the following partial differential equations, identify the equation as parabolic, elliptic or hyperbolic and find a transform to put the system in a standard form.

(a) $2u_{xx} + 4u_{xt} + 2u_{tt} = 0$
Here $A = 2$, $B = 2$ and $C = 2$ so $B^2 - AC = 0$ and the equation is parabolic. So using my notation from class, $\frac{a}{b} = -\frac{B}{C} = -1$. We can set $a = 1$, $b = -1$. We must choose c and d so that $ad - bc \neq 0$. So lets take $c = d = 1$. So the transform is

$$\begin{aligned} \xi &= x - y\,, \\ \eta &= x + y\,. \end{aligned}$$

(b) $8u_{xx} + 10u_{xy} + 2u_{yy} = 0$
Here $A = 8$, $B = 5$ and $C = 2$, so $B^2 - AC = 9$ so the equation is hyperbolic. So we have $\frac{a}{b} = -\frac{1}{4}$ and $\frac{c}{d} = -1$. We choose the coordinates,

$$\begin{aligned} \xi &= x - 4y\,, \\ \eta &= x - y\,. \end{aligned}$$

(c) $10u_{xx} + 12u_{xy} + 4u_{yy} = 0$
Now $A = 10$, $B = 6$ and $C = 4$, so $B^2 - AC = -4$ and the system is elliptic. So we will need,

$$\begin{aligned} a &= \frac{6c + 4d}{2}\,, \\ b &= -\frac{6c + 10d}{2}\,. \end{aligned}$$

So we can pick $c = d = 1$. This results in $a = 5$ and $b = -8$. We note that $ad - bc = 13 \neq 0$. So,

$$\begin{aligned} \xi &= 5x - 8y\,, \\ \eta &= x + y\,. \end{aligned}$$

5. If $u(x,t)$ satisfies $u_{tt} = u_{xx}$, prove the identity

$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h)$$

for all x, t, h and k. Sketch the quadrilateral Q whose vertices are the arguments of the identity.
The general solutions for u is given by $u = f(x+t) + g(x-t)$ for some f and g. The Right hand side of the identity is given by:

$$RHS = f(x+t+h+k) + g(x-t+h-k) + f(x+t-h-k) + g(x-t-h+k)$$

and the left hand side is given by:

$$LHS = f(x+t+h+k) + g(x-t+k-h) + f(x+t-k-h) + g(x-t-k+h)$$

The two sides are equal and we are done. I will skip the sketch.