A FOUR DIMENSIONAL CONTINUOUS WAVELET TRANSFORM

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Abstract. Higher dimensional analogs of the classical continuous wavelet transform are developed for Euclidean spaces whose dimension is a perfect square. For a positive integer n, the space of all $n \times n$ real matrices can be identified with $\mathbb{R}^{n^2}$ as an additive abelian group. The group of invertible $n \times n$ real matrices naturally acts on this abelian group by matrix multiplication. The resulting semidirect product group forms a distinguished group of affine transformations of $\mathbb{R}^{n^2}$ which may be viewed as a natural generalization of the full group of affine transformations of $\mathbb{R}$ whose unique square-integrable representation underlies the classical one-dimensional continuous wavelet transform. A continuous wavelet transform for $\mathbb{R}^{n^2}$ is derived and its specific details are worked out for $\mathbb{R}^4$ resulting in a 4D continuous wavelet transform. This transform is discretized by introducing a geometrically intuitive tiling system for $\mathbb{R}^4$ and constructing discrete frames based on this tiling system.

1. INTRODUCTION

As computational power increases, researchers in a wide variety of disciplines find value in imaging three dimensional structures in motion. Sample areas where the resulting 4D data must be processed are dynamic NMR [20], geophysics [18], medical imaging [21], ultrasound images of a fetus in motion [5], and computer graphics [23]. Our goal in this paper is to initiate the detailed investigation of a four dimensional continuous wavelet transform that is based on the general approach developed in [2]. Besides the potential that we will eventually be able to provide useful computational techniques for 4D data analysis, we are interested in investigating this particular transformation because it is a direct generalization of the now classical continuous wavelet transform on $\mathbb{R}$. We identify $\mathbb{R}^4$ with the group of 2×2 real matrices, let H denote the group of invertible 2×2 real matrices, and consider the elements of H as acting on $\mathbb{R}^4$ through matrix multiplication. Combining translations in $\mathbb{R}^4$ with these dilations coming from H, we get an eight dimensional Lie group which shares some important properties with the group that underlies the classical continuous wavelet transform. In particular, this eight dimensional Lie group has a unique (up to unitary equivalence) square-integrable irreducible unitary representation. The reader is directed to [1] for a general overview of continuous wavelet-like transforms and their applications to a variety of physical situations.

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In [11], it was recognized that the reconstruction identity that forms the basis for the continuous wavelet transform on $\mathbb{R}$ can be interpreted as a special case of a generalized orthogonality relation for coefficient functions of a square-integrable representation of a locally compact group. In the case of the continuous wavelet transform, the group in question is the group of affine transformations of $\mathbb{R}$ and the combination of translations and dilations provide the square-integrable representation. See [12] for a discussion of this view of the continuous wavelet transform. In [2], a general framework for the development of higher dimension continuous wavelet transforms was investigated. Essentially, if one has a locally compact group H acting on $\mathbb{R}^n$ in such a manner that there exists an open subset O in $\mathbb{R}^n$ (actually the Pontryagin dual version of $\mathbb{R}^n$) so that H acts freely and transitively on O, then there exists an associated continuous wavelet transform theory. See [10] for a comprehensive investigation of an abstract approach to continuous wavelet transforms. The two dimensional continuous shearlet transform [14] can be viewed in this manner. See also [19] for similar 2D transforms derived from extending the three dimensional Heisenberg group by dilations. In [3], a higher dimensional version of the shearlet transform is proposed. It shares the anisotropic features of the 2D shearlet transform that are useful in many situations. Note that the transform presented in this paper is decidedly isotropic. The value of square-integrability to the application of a continuous wavelet transform to characterizing smoothness spaces of functions is discussed in [4].

Because the 4D transform we introduce here involves using eight variables to move a potential wavelet around, we provide the admissibility condition and the reconstruction formula in both the detailed form that will be necessary for applications with the precise role played by all variables evident and in an abstract form where the underlying group provides elegance and ease of proofs. In the more abstract form, it is just as easy to work with $n \times n$ real matrices, which results in a continuous wavelet transform on $\mathbb{R}^{n^2}$. This transform reduces to the classical 1D transform when $n = 1$ and our desired 4D transform when $n = 2$. After stating the concrete 4D case as our main theorem in Section 2, the basic notation and properties we need are collected in Sections 2 and 3 and the continuous wavelet transform on $\mathbb{R}^{n^2}$ is given in Section 4 with the proof of Theorem 2.1 following immediately. In Section 5, as a first step towards a useful discretization of this transform, we introduce the concept of a tiling system in an orbit of a locally compact group and construct an explicit tiling system for a particular $GL_2(\mathbb{R})$ orbit which is then used to construct a discrete frame in $\mathbb{R}^4$, in Section 6.

2. THE FOUR DIMENSIONAL TRANSFORM

Before describing the somewhat abstract background necessary, it may be useful to present the wavelet condition and reconstruction formula as it appears in the four dimensional case.

For $\psi \in L^2(\mathbb{R}^4)$ and parameters $x = (x_1, x_2, x_3, x_4)$, $h = (h_1, h_2, h_3, h_4) \in \mathbb{R}^4$, define $\psi_{x,h} \in L^2(\mathbb{R}^4)$ by, for $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$,

$$\psi_{x,h}(y_1, y_2, y_3, y_4) = \frac{1}{|h_1h_4 - h_2h_3|}\psi\left(b_1, b_2, b_3, b_4\right),$$

where

$$b_j = \frac{h_4}{h_1h_4 - h_2h_3}\left(y_j - x_j\right) - \frac{h_2}{h_1h_4 - h_2h_3}\left(y_{j+2} - x_{j+2}\right), \text{ when } j \in \{1, 2\},$$

$$b_j = \frac{h_1}{h_1h_4 - h_2h_3}\,(y_j - x_j) - \frac{h_3}{h_1h_4 - h_2h_3}\,(y_{j-2} - x_{j-2})\,,\ \text{when } j \in \{3,4\}.$$

Our main theorem characterizes those ψ which can serve as a wavelet in the 4D continuous wavelet transform.

Theorem 2.1. *Let $\psi \in L^2(\mathbb{R}^4)$. If*

$$\int_{\mathbb{R}^4} \left|\widehat{\psi}(h_1,h_2,h_3,h_4)\right|^2 \frac{dh_1dh_2dh_3dh_4}{|h_1h_4 - h_2h_3|^2} = 1, \tag{1}$$

then, for any $f \in L^2(\mathbb{R}^4)$,

$$f = \int_{\mathbb{R}^4}\int_{\mathbb{R}^4} \langle f, \psi_{x,h}\rangle\, \psi_{x,h}\ \frac{dx_1\cdots dx_4\, dh_1\cdots dh_4}{|h_1h_4 - h_2h_3|^4}, \tag{2}$$

weakly in $L^2(\mathbb{R}^4)$.

Conversely, if (2) *holds for every $f \in L^2(\mathbb{R}^4)$, then ψ satisfies* (1).

Theorem 2.1 will follow immediately from the general formulation in Section 4.

3. NOTATION AND DEFINITIONS

If X is a locally compact space, $C_0(X)$ denotes the Banach space of continuous complex-valued functions on X which vanish at infinity, equipped with the supremum norm, and $C_c(X)$ denotes the dense subspace consisting of the continuous functions with compact support. If there is a distinguished regular Borel measure μ on X, then $L^p(X) = L^p(X,\mu)$ denotes the usual Lebesgue space, for $1 \le p \le \infty$. Note that $C_c(X)$ is dense in $L^p(X)$, for $1 \le p < \infty$. The Hilbert space structure of $L^2(X)$ has inner product $\langle f,g\rangle = \int_X f(x)\overline{g(x)}\,d\mu(x)$, for $f,g \in L^2(X)$.

Let n be a positive integer. The set of $n\times n$ real matrices $M_n(\mathbb{R})$ is an algebra over $\mathbb{R}$ when equipped with matrix addition, matrix multiplication and multiplication by scalars. It is a topological algebra when given the topology of $\mathbb{R}^{n^2}$ under the obvious identification. For $x \in M_n(\mathbb{R})$, the determinant of x, $\det(x)$, is a polynomial in the coordinates of x and $GL_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) :\ \det(x) \neq 0\}$ is a dense open subset of $M_n(\mathbb{R})$. When considering $M_n(\mathbb{R})$ as an abelian group under addition, we will denote it as A. Note A is just $\mathbb{R}^{n^2}$. We also introduce the notation H for $GL_n(\mathbb{R})$, considered as a locally compact group which naturally acts on A by matrix multiplication. When $n = 1$, $A = \mathbb{R}$ and $H = \mathbb{R}^* = \{h \in \mathbb{R} :\ h \neq 0\}$. The reader will notice the classical theory of the continuous wavelet transform on $\mathbb{R}$ in the following.

For $x \in A$ and $h \in H$ let $[x,h]$ denote the affine transformation of A given by $[x,h]z = hz + x$, for $z \in A$, where hz is the simple product of the matrices h and z. By composing transformations, we get a product operation,

$$[x,h][y,k] = [x + hy, hk]. \tag{3}$$

Let $A \rtimes H = \{[x,h] : x \in A, h \in H\}$ equipped with the product (3). Then $A \rtimes H$ is a locally compact group when given the product topology. If I denotes the $n \times n$ identity matrix, then $[0,I]$ is the identity in $A \rtimes H$ and $[x,h]^{-1} = [-h^{-1}x, h^{-1}]$, for $[x,h] \in A \rtimes H$.

We equip A with Lebesgue measure under the identification with $\mathbb{R}^{n^2}$ and $\int_A f(x)dx$ will denote Lebesgue integration. For $h \in H$, let $\delta(h) = |\det(h)|^n$. Then for any integrable function g on A,

$$\int_A g(x)dx = \delta(h) \int_A g(hx)dx. \tag{4}$$

We also need integration over H. Any locally compact group G carries a regular Borel measure which is invariant under left translation called left Haar measure. This measure is unique up to a constant multiple. This measure can be specified by the positive linear functional it defines on $C_c(G)$. See [13] or [9] for the properties of Haar measure and the Haar integral. We will denote the integral with respect to the left Haar measure on H by $\int_H g(h)dh$ for any function $g \in C_c(H)$ or for any function g for which this integral makes sense. In [13] one finds that, since $H = GL_n(\mathbb{R})$,

$$\int_H g(h)dh = \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \dots & h_{nn} \end{pmatrix} \frac{dh_{11}dh_{12}\dots dh_{1n}dh_{21}\dots dh_{n1}\dots dh_{nn}}{|\det(h)|^n}, \tag{5}$$

where $h = (h_{ij})_{i,j=1}^n$ is a generic element of H. It turns out that left Haar measure on $GL_n(\mathbb{R})$ is also right invariant and, hence, inversion invariant. That is, for any $h' \in H$ and $g \in C_c(H)$,

$$\int_H g(h'h)\, dh = \int_H g(hh')\, dh = \int_H g(h^{-1})\, dh = \int_H g(h)\, dh.$$

Now we can describe left Haar integration on $A \rtimes H$. For $f \in C_c(A \rtimes H)$,

$$\int\limits_{A \rtimes H} f([x,h])\, d[x,h] = \int_H \int_A f([x,h])\delta(h)^{-1} dx\, dh. \tag{6}$$

It is a routine calculation to show that the integral given on the right hand side of (6) is invariant under left translations.

For the purpose of Fourier analysis on $\mathbb{R}^{n^2}$, identified with A, there are many ways to pair A with $\widehat{A}$, the group of characters on A. To exploit the notational advantage of matrix multiplication we chose the following identification. For $b = (b_{ij})_{i,j=1}^n \in A$, define $\chi_b \in \widehat{A}$ by

$$\chi_b(x) = e^{2\pi i \operatorname{tr}(bx)}, \text{ for } x \in A. \tag{7}$$

We have, $\widehat{A} = \{\chi_b : \ b \in A\}$. Thus, $\widehat{A}$ can also be identified with $\mathbb{R}^{n^2}$ and Haar integration on $\widehat{A}$ is simply the Lebesgue integral. That is

$$\int_{\widehat{A}} g(\chi)\, d\chi = \int_{\mathbb{R}^{n^2}} g(\chi_b)\, db = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(\chi_{(b_{11},\dots,b_{nn})})\, db_{11}\dots db_{nn},$$

where we are thinking of b as an n^2-vector. For $f \in L^1(A)$, the Fourier transform $\widehat{f}\colon \widehat{A} \to \mathbb{C}$ is given by $\widehat{f}(\chi) = \int_A f(x)\chi(x)dx$, for all $\chi \in \widehat{A}$. Then $\widehat{f} \in C_0(\widehat{A})$. For $f \in L^1(A) \cap L^2(A)$, $\widehat{f} \in L^2(\widehat{A})$ and $\|\widehat{f}\|_2 = \|f\|_2$. There is a unitary map $\mathcal{P}\colon L^2(A) \to L^2(\widehat{A})$, the Plancherel transform, such that $\mathcal{P}f = \widehat{f}$, for all $f \in L^1(A) \cap L^2(A)$.

The action H on A determines an action of H on $\widehat{A}$ by, for $h \in H,\ \chi \in \widehat{A},\ (h \cdot \chi)(x) = \chi(h^{-1} \cdot x)$, for all $x \in A$. Then, $h \cdot \chi_b = \chi_{bh^{-1}}$, for $b \in A$ and $h \in H$. This action scales Lebesgue measure, so that, for any $\xi \in C_c(\widehat{A})$,

$$\int_{\widehat{A}} \xi(\chi)\, d\chi = \delta(h)^{-1} \int_{\widehat{A}} \xi(h \cdot \chi)\, d\chi, \tag{8}$$

which can be verified by direct computation. As usual, (8) holds for any function ξ for which the integrals make sense. There are special features of the H-orbit structure in $\widehat{A}$ that are critical to the existence of a continuous wavelet transform strongly connected to the group structure. Let $O = \{h \cdot \chi_I : h \in H\}$. We gather the properties of this H-orbit together in a proposition.

Proposition 3.1. *With the above notation,*

(i) $O = \{\chi_{h^{-1}} : h \in H\} = \{\chi_h : h \in H\}$.
(ii) *O is a dense open subset of $\widehat{A}$.*
(iii) *$\widehat{A} \setminus O$ is a null set with respect to Lebesgue measure on $\widehat{A}$.*
(iv) *The map $h \to h \cdot \chi_I$ is a homeomorphism of H with O.*
(v) *For any $\xi \in C_c(O)$, $\int_{\widehat{A}} \xi(\chi)\, d\chi = \int_H \xi(h \cdot \chi_I)\delta(h)^{-1} dh$.*
(vi) *For any $\xi \in L^2(\widehat{A})$ and any two elements $\chi, \omega \in O$, $\int_H |\xi(h^{-1} \cdot \chi)|^2 dh = \int_H |\xi(h^{-1} \cdot \omega)|^2 dh$.*

Proof. Assertion (i) is simply because H is a group. Assertion (iii) follows from (ii) and for (ii), we note that $GL_n(\mathbb{R})$ is a dense open subset of $M_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ maps onto O under the parametrization of $\widehat{A}$ given by (7). Assertion (iv) is obvious.

For (v), let $\xi \in C_c(O)$. Recall $\delta(h) = |\det(h)|^n$, for $h \in H$, and that the Haar integral on H, given in (5), is inversion invariant. Then

$$\int_H \xi(h \cdot \chi_I)\delta(h)^{-1} dh = \int_H \xi(\chi_{h^{-1}})\delta(h^{-1}) dh = \int_H \xi(\chi_h)\delta(h) dh = \int_{\mathbb{R}^{n^2}} \xi(\chi_b) db = \int_{\widehat{A}} \xi(\chi) d\chi.$$

Now, let $\xi \in L^2(\widehat{A}), \chi, \omega \in O$. There exists a $h' \in H$ such that $\omega = h' \cdot \chi$, so $\chi = h'^{-1} \cdot \omega$. Then, left invariance of the Haar integral on H implies

$$\int_H |\xi(h^{-1} \cdot \chi)|^2 dh = \int_H |\xi(h^{-1} \cdot h'^{-1} \cdot \omega)|^2 dh = \int_H |\xi((h'h)^{-1} \cdot \omega)|^2 dh = \int_H |\xi(h^{-1} \cdot \omega)|^2 dh.$$

Thus, (vi) holds. □

4. A SQUARE-INTEGRABLE IRREDUCIBLE REPRESENTATION

In this section, we provide three equivalent versions of the distinguished irreducible representation of $A \rtimes H$ that underlies the continuous wavelet transform introduced in the next section. Let G be a locally compact group and $\mathcal{H}$ a Hilbert space. Let $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. A representation of G on $\mathcal{H}$ is a homomorphism $\sigma \colon G \to \mathcal{U}(\mathcal{H})$ which is continuous if $\mathcal{U}(\mathcal{H})$ carries the weak operator topology. The representation σ is called irreducible if $\{0\}$ and $\mathcal{H}$ are the only closed subspaces of $\mathcal{H}$ invariant under σ. For any $\xi, \eta \in \mathcal{H}$, define $\varphi^\sigma_{\xi,\eta}(x) = \langle \eta, \sigma(x)\xi \rangle$, for all $x \in G$. The requirement that σ be continuous when $\mathcal{U}(\mathcal{H})$ is equipped with the weak operator topology means $\varphi^\sigma_{\xi,\eta}$ is a continuous function on G. The representation σ is irreducible if and only if $\varphi^\sigma_{\xi,\eta} = 0$ implies at least one of ξ or η is 0. If σ_1 and σ_2 are two representations of G on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, we say σ_1 is (unitarily) equivalent to σ_2 if there exists a unitary transformation $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\sigma_1(x) = \sigma_2(x)U$, for all $x \in G$. Let $\widehat{G}$ denote the space of equivalence classes of irreducible representations of G. An introduction to the representation theory of locally compact groups can be found in [9].

An irreducible representation of G, say σ acting on $\mathcal{H}$, is called square–integrable if there exist nonzero $\xi, \eta \in \mathcal{H}$ such that $\varphi^{\sigma}_{\xi,\eta} \in L^2(G)$. See Chapter 14 of [6] for the basic theory of square–integrable representations. With $\xi \in \mathcal{H} \setminus \{0\}$ fixed, if there exists one nonzero $\eta' \in \mathcal{H}$ with $\varphi^{\sigma}_{\xi,\eta'} \in L^2(G)$, then $\varphi^{\sigma}_{\xi,\eta} \in L^2(G)$ for any $\eta \in \mathcal{H}$. Such a vector ξ is called admissible and the set of admissible vectors is dense in $\mathcal{H}$.

For the group $A \rtimes H$, we will give explicit definitions of three representations ρ, π and τ of $A \rtimes H$, all of which turn out to be mutually equivalent.

The Hilbert space of ρ is $L^2(A)$ and ρ is the natural combination of translation on A with dilation by members of H. For $[x,h] \in A \rtimes H$, define $\rho[x,h]$ on $L^2(A)$ by, for $g \in L^2(A)$,

$$\rho[x,h]g(y) = \delta(h)^{-1/2} g(h^{-1}(y-x)), \tag{9}$$

for all $y \in A$.

The Hilbert space of π is $L^2(\widehat{A})$. For $[x,h] \in A \rtimes H$ and $\xi \in L^2(\widehat{A})$,

$$\pi[x,h]\xi(\chi) = \delta(h)^{1/2} \chi(x) \xi(h^{-1} \cdot \chi), \tag{10}$$

for all $\chi \in \widehat{A}$.

The Hilbert space of τ is $L^2(H)$. For $[x,h] \in A \rtimes H$ and $f \in L^2(H)$,

$$\tau[x,h]f(k) = (k \cdot \chi_I)(x) f(h^{-1} \cdot k), \tag{11}$$

for all $k \in H$. We leave the routine checks that each of ρ, π and τ satisfies all the properties of a representation of $A \rtimes H$ to the reader.

Proposition 4.1. *The three representations ρ, π and τ, defined in* (9), (10) *and* (11) *are pairwise mutually equivalent representations of $A \rtimes H$.*

Proof. It is a simple matter to show that

$$\pi[x,h] = \mathcal{P}\rho[x,h]\mathcal{P}^{-1},$$

for all $[x,h] \in A \rtimes H$, where $\mathcal{P}\colon L^2(A) \to L^2(\widehat{A})$ is the Plancherel transform. Since $\mathcal{P}$ is a unitary map, π is equivalent to ρ.

By Proposition 3.1 (ii) and (iii), $C_c(O)$ is dense in $L^2(\widehat{A})$. For $\xi \in C_c(O)$, define $W\xi$ on H by

$$W\xi(h) = \delta(h)^{-1/2} \xi(h \cdot \chi_I),$$

for $h \in H$. Then W is a linear map of $C_c(O)$ onto $C_c(H)$ by Proposition 3.1 (iv). Moreover, $\|W\xi\|_{L^2(H)} = \|\xi\|_{L^2(\widehat{A})}$ by Proposition 3.1 (v). Thus, W extends to a unitary map of $L^2(\widehat{A})$ onto $L^2(H)$. One directly computes that $\tau[x,h] = W\pi[x,h]W^{-1}$, for all $[x,h] \in A \rtimes H$.

Thus, any pair from $\{\rho, \pi, \tau\}$ are equivalent. □

Proposition 4.2. *Each of ρ, π or τ is a square–integrable irreducible representation of $A \rtimes H$.*

Proof. This is Theorem 1 in [2]. However, the insight of Proposition 3.1 allows us to avoid introducing the Radon-Nikodym derivative that moves integration over H to Lebesgue integration on the H-orbit.

Since all three representations are equivalent, we will work with π. Let $\xi, \eta \in L^2(\widehat{A})$. Note that $\int\limits_H \left|\xi\left(h^{-1}\cdot\chi\right)\right|^2 dh$ is independent of $\chi \in O$ by (vi) of Proposition 3.1. Moreover, $h^{-1}\cdot\chi_I = \chi_h$, for all $h \in H$, so $\int\limits_H \left|\xi\left(h^{-1}\cdot\chi\right)\right|^2 dh = \int\limits_H \left|\xi\left(\chi_h\right)\right|^2 dh$, for almost all $\chi \in \widehat{A}$. The following calculation is similar to (2.13) in [2]. Let $\omega_h(\chi) = \eta(\chi)\overline{\xi}(h^{-1}\cdot\chi)$, for almost all $\chi \in \widehat{A}$ and let $\omega_h^{\vee}$ denote its inverse Fourier transform. Then

$$\begin{aligned}
\int\limits_{A\rtimes H} \left|\varphi^{\pi}_{\xi,\eta}([x,h])\right|^2 d[x,h] &= \int_H \int_A \left|\int_{\widehat{A}} \eta(\chi)\delta(h)^{1/2}\,\overline{\chi(x)}\,\overline{\xi}(h^{-1}\cdot\chi)\,d\chi\right|^2 \delta(h)^{-1} dx\,dh \\
&= \int_H \int_A \left|\int_{\widehat{A}} \omega_h(\chi)\,\overline{\chi(x)}\;d\chi\right|^2 dx\,dh = \int_H \int_A |\omega_h^{\vee}(x)|^2\,dx\,dh \\
&= \int_H \int_{\widehat{A}} |\omega_h(\chi)|^2\,d\chi\,dh = \int_{\widehat{A}} |\eta(\chi)|^2 \int_H |\xi(h^{-1}\cdot\chi)|^2 d\chi\,dh \\
&= \|\eta\|_2^2 \int_H \left|\xi\left(\chi_h\right)\right|^2 dh.
\end{aligned} \tag{12}$$

If $\xi \neq 0$, then $\int_H \left|\xi\left(\chi_h\right)\right|^2 dh \neq 0$ by Proposition 3.1 (iv). Thus, $\varphi^{\pi}_{\xi,\eta} \neq 0$ if ξ and η are both nonzero. Therefore, π is irreducible.

Moreover, if $\xi \in C_c(O)$, then $\int_H \left|\xi\left(\chi_h\right)\right|^2 dh < \infty$. So $\int\limits_{A\rtimes H} |\varphi^{\pi}_{\xi,\eta}([x,h])|^2 d[x,h] < \infty$, for any $\xi \in C_c(O)$, and $\eta \in L^2(\widehat{A})$. Thus, π is square-integrable. □

Remark 4.3. *Using the equivalence of ρ with π,* (12) *says that, for $f, g \in L^2(A)$,*

$$\int\limits_{A\rtimes H} \left|\langle f, \rho[x,h]g\rangle\right|^2 d[x,h] = \|f\|_2^2 \int\limits_H \left|\widehat{g}\left(\chi_h\right)\right|^2 dh. \tag{13}$$

5. THE GENERAL CONTINUOUS WAVELET TRANSFORM

As usual, (13) forms the basis of a continuous wavelet transform (CWT) for which we now provide the details.

Definition 5.1. *A function $\psi \in L^2(A)$ is said to satisfy the wavelet condition if*

$$\int_H |\widehat{\psi}(\chi_h)|^2 dh = 1. \tag{14}$$

If $\psi \in L^2(A)$ is a fixed function satisfying the wavelet condition, define the linear transformation $V_\psi \colon L^2(A) \to L^2(A \rtimes H)$ by

$$V_\psi f[x,h] = \langle f, \rho[x,h]\psi\rangle\,, \tag{15}$$

for $f \in L^2(A), [x,h] \in A \rtimes H$. By (13), V_ψ is an isometry of $L^2(A)$ into $L^2(A \rtimes H)$. Thus, V_ψ is a unitary map onto its range. This implies that

$$\langle f, g\rangle = \int\limits_{A\rtimes H} \langle f, \rho[x,h]\psi\rangle\,\langle \rho[x,h]\psi, g\rangle\,d[x,h],$$

for any $f, g \in L^2(A)$. For a fixed f, g is an arbitrary element of $L^2(A)$, resulting in the forward implication in the following proposition. The reverse implication is immediate from (13).

Proposition 5.2. *Let $\psi \in L^2(A)$. Then ψ satisfies the wavelet condition if and only if, for any $f \in L^2(A)$,*

$$f = \int_{A \rtimes H} \langle f, \rho[x,h]\psi \rangle \, \rho[x,h]\psi \, d[x,h] \tag{16}$$

weakly in $L^2(A)$.

To put Proposition 5.2 more in the style of wavelet analysis, introduce the notation

$$\psi_{x,h}(y) = \rho[x,h]\psi(y) = |\det(h)|^{-n/2}\psi(h^{-1}(y-x)),$$

for $y, x \in M_n(\mathbb{R})$ and $h \in GL_n(\mathbb{R})$. The wavelet condition (14) becomes

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left| \widehat{\psi}(h_{11}, h_{12} \cdots, h_{nn}) \right|^2 \frac{dh_{11} dh_{12} \cdots dh_{nn}}{|\det(h)|^n} = 1, \tag{17}$$

where $h = (h_{ij})_{i,j=1}^n$. Proposition 5.2 says $\psi \in L^2(\mathbb{R}^{n^2})$ satisfies (17) if and only if, for any $f \in L^2(\mathbb{R}^{n^2})$

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \langle f, \psi_{x,h} \rangle \, \psi_{x,h} \, dx_{11} dx_{12} \cdots dx_{nn} \right) \frac{dh_{11} dh_{12} \cdots dh_{nn}}{|\det(h)|^{2n}}, \tag{18}$$

weakly in $L^2(\mathbb{R}^{n^2})$, where both x and h in $\mathbb{R}^{n^2}$ are indexed as if they are arranged as a square matrix.

Letting $n = 2$, writing $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$ and computing the matrix inversion and products yields Theorem 2.1.

6. A discrete frame

In this section, we construct a discrete frame in $L^2(\mathbb{R}^4)$ based on the reconstruction formula of Theorem 2.1. For this, we follow the method presented in [2], modified to take advantage of detailed structural knowledge of $H = \mathrm{GL}_2(\mathbb{R})$.

We begin with a decomposition for $\mathrm{GL}_2(\mathbb{R})$, which is an extension of the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$ (see [15]). To be self-contained, we include a proof in this article. Let $K = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\}$ denote the compact subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of rotations and define two abelian subgroups D and N by

$$D = \left\{ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} : r \in \mathbb{R}^+, s \in \mathbb{R}^* \right\}, \quad \text{and} \quad N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\},$$

where $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$ and $\mathbb{R}^* = \{t \in \mathbb{R} : t \neq 0\}$.

Proposition 6.1. *Every element of $\mathrm{GL}_2(\mathbb{R})$ can be uniquely decomposed as an ordered product of elements in K, D, and N. That is, $\mathrm{GL}_2(\mathbb{R}) = KDN$.*

Proof. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ be given. We will find the unique solution to the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}. \tag{19}$$

Indeed, multiplying both sides of the equation by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and computing the norms, we get $r = \sqrt{a^2 + c^2}$, which is greater than zero as $ad - bc \neq 0$. Moreover, we can write Equation (19) as

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & rx \\ 0 & s \end{pmatrix},$$

which implies that

$$\begin{cases} r = a\cos\theta + c\sin\theta \\ 0 = -a\sin\theta + c\cos\theta \\ rx = b\cos\theta + d\sin\theta \\ s = -b\sin\theta + d\cos\theta \end{cases}.$$

Thus,

$$\theta = \begin{cases} \pi/2 & \text{if } a = 0 \text{ and } c > 0 \\ 3\pi/2 & \text{if } a = 0 \text{ and } c < 0 \\ 0 & \text{if } c = 0 \text{ and } a > 0 \\ \pi & \text{if } c = 0 \text{ and } a < 0 \\ \tan^{-1}(c/a) & \text{if } a \neq 0, c \neq 0, \text{ and } a\cos(\tan^{-1}(c/a)) + c\sin(\tan^{-1}(c/a)) > 0 \\ \tan^{-1}(c/a) + \pi & \text{if } a \neq 0, c \neq 0, \text{ and } a\cos(\tan^{-1}(c/a)) + c\sin(\tan^{-1}(c/a)) < 0, \end{cases} \quad (20)$$

where $\tan^{-1}$ is the inverse function of tan in the interval $[0, \pi) \setminus \{\frac{\pi}{2}\}$. It is easy to check that θ defined in Equation (20) and $r = \sqrt{a^2 + c^2}$ satisfy $r = a\cos\theta + c\sin\theta$. Finally, we have $x = \frac{b\cos\theta + d\sin\theta}{\sqrt{a^2+c^2}}$ and $s = -b\sin\theta + d\cos\theta$. Clearly, $s \neq 0$. From the above discussion, it is clear that this solution is unique. ☐

Let $\mathrm{GL}_2^+(\mathbb{R})$ (respectively $\mathrm{GL}_2^-(\mathbb{R})$) denote the subset of elements of $\mathrm{GL}_2(\mathbb{R})$ with positive (respectively negative) determinants. Note that $\mathrm{GL}_2^+(\mathbb{R})$ is the connected component of the identity in $\mathrm{GL}_2(\mathbb{R})$ and, as such, is a closed normal subgroup. Define the following additional three closed subgroups of $GL_2(\mathbb{R})$:

$$B = \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^+,\ x \in \mathbb{R} \right\},$$

$$T = \left\{ \begin{pmatrix} r & x \\ 0 & s \end{pmatrix} : r \in \mathbb{R}^+, s \in \mathbb{R}^*,\ x \in \mathbb{R} \right\} = DN,$$

and

$$T^+ = \left\{ \begin{pmatrix} r & x \\ 0 & s \end{pmatrix} : r, s \in \mathbb{R}^+,\ x \in \mathbb{R} \right\}.$$

Since, for $r, s \in \mathbb{R}^+,\ x \in \mathbb{R}$, $\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} = \sqrt{rs} \begin{pmatrix} \sqrt{r/s} & x/\sqrt{rs} \\ 0 & \sqrt{s/r} \end{pmatrix}$, we have $T^+ = \mathbb{R}^+ B$. Let $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then T^+ and uT^+ are the two cosets of T^+ in T. So $T = T^+ \cup uT^+$ and by Proposition 6.1, $GL_2(\mathbb{R}) = KT$ and $K \cap T$ consists of the identity only.

Where it is notationally convenient, we continue to use A for $\mathbb{R}^4$ with elements arranged as 2×2 matrices, $H = GL_2(\mathbb{R})$, and $A \rtimes H$, and O as defined in Section 3. Recall that O is an H-orbit in $\widehat{A}$ and, by Proposition 3.1(i), $O = \{\chi_h : h \in H\}$.

Definition 6.2. *Let P be a countable subset of H, and let F be a measurable relatively compact subset of O. The pair (F, P) is called a* tiling system for the orbit O *if the following two conditions are satisfied:*

(i) $p \cdot F \bigcap q \cdot F = \emptyset$ *for every pair $p \neq q$ in P.*
(ii) $O = \bigcup \{p \cdot F : p \in P\}$.

Let (F, P) be a tiling system for O. For each $p \in P$, let $L^2(p \cdot F)$ denote the closed subspace of $L^2(\widehat{A})$ consisting of functions that are zero almost everywhere on $\widehat{A} \setminus p \cdot F$. Noting that O is a co-null subset of $\widehat{A}$, we have that

$$L^2(\widehat{A}) = \sum_{p \in P} \oplus L^2(p \cdot F).$$

Proposition 6.3. *Let* $P = \left\{ \begin{pmatrix} 2^{l+k} & 2^{l+k}j \\ 0 & 2^{l-k} \end{pmatrix} : k, l, j \in \mathbb{Z} \right\}$,

$$E = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha w & \alpha y \\ 0 & \pm\alpha w^{-1} \end{pmatrix} : 0 \leq \theta < 2\pi,\ 1 \leq w < 2,\ 1 \leq \alpha < 2,\ 0 \leq y < w \right\}$$

and $F = \{\chi_b : b \in E\}$. Then (F, P) forms a tiling system for O.

Proof. For every subset $L \subseteq H$, let O_L denote the set $\{\chi_h : h \in L\}$. Clearly $p \cdot O_L = O_{Lp^{-1}}$ for every $L \subseteq H$ and $p \in H$. Thus, to find a tiling system (O_E, P) for O it is enough to find a countable subset P of H and a relatively compact subset E of H such that

(i) $Ep^{-1} \bigcap Eq^{-1} = \emptyset$ for all $p, q \in P, p \neq q$,
(ii) $H = \bigcup_{p \in P} Ep^{-1}$.

We will use the decomposition stated in Proposition 6.1 to construct a tiling system of O in three steps.

Claim 6.4. *Let* $P_1 = \left\{ \begin{pmatrix} 2^k & 2^k j \\ 0 & 2^{-k} \end{pmatrix} : k, j \in \mathbb{Z} \right\}$ *and* $E_1 = \left\{ \begin{pmatrix} w & y \\ 0 & w^{-1} \end{pmatrix} : 1 \leq w < 2, 0 \leq y < w \right\}$.
Then E_1 is a relatively compact subset of H such that

(i) $E_1 p^{-1} \bigcap E_1 q^{-1} = \emptyset$ *for every $p \neq q$ in P_1.*
(ii) $B = \bigcup_{p \in P_1} E_1 p^{-1}$.

Proof of claim. Clearly E_1 is relatively compact. To prove (i) and (ii), we show that for every $\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}$ in B, the equation

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} w & y \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} 2^{-k} & -2^k j \\ 0 & 2^k \end{pmatrix}$$

has a unique solution with constraints $k, j \in \mathbb{Z}$, $1 \leq w < 2$, and $0 \leq y < w$. Indeed, $k = -\lfloor \log_2 a \rfloor$ and $w = a 2^{-\lfloor \log_2 a \rfloor}$ are uniquely determined. Finally, since $\mathbb{R} = \cup_{i \in \mathbb{Z}} [-wi, -w(i-1))$, there exist unique $j \in \mathbb{Z}$ and $y \in [0, w)$ such that $x 2^{-k} = -wj + y$. Thus the above matrix equation has a unique solution, which finishes the proof of the claim.

Let $E_2 = JE_1$, where $J = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : 1 \leq \alpha < 2 \right\}$. Note that $\mathbb{R}^+ = \bigcup_{l \in \mathbb{Z}} 2^l [1, 2)$, which is a disjoint union.

Claim 6.5. *With P as in Proposition 6.3,*

(i) $E_2 p^{-1} \cap E_2 q^{-1} = \emptyset$ *for every $p \neq q$ in P.*

(ii) $T^+ = \bigcup_{p\in P} E_2 p^{-1}$.

Proof of claim. Note that $E_2 = \left\{ \begin{pmatrix} \alpha\omega & \alpha y \\ 0 & \alpha\omega^{-1} \end{pmatrix} : \ 1 \le \alpha < 2,\ 1 \le \omega < 2,\ 0 \le y < \omega \right\}$.
We need to show that, for any $\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} \in T^+$, the equation

$$\begin{pmatrix} r & x \\ 0 & s \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \omega & y \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} 2^{-l-k} & -2^{-l+k}j \\ 0 & 2^{-l+k} \end{pmatrix} = \alpha 2^{-l} \begin{pmatrix} \omega & y \\ 0 & \omega^{-1} \end{pmatrix} \begin{pmatrix} 2^{-k} & -2^k j \\ 0 & 2^k \end{pmatrix}$$

has a unique solution with, $1 \le \alpha < 2,\ 1 \le \omega < 2,\ 0 \le y < \omega,\ l,k,j \in \mathbb{Z}$. By taking determinants we get $\sqrt{rs} = \alpha 2^{-l}$ which implies $l = -\lfloor \log_2 \sqrt{rs} \rfloor$ and $\alpha = \sqrt{rs}2^{-l}$. The unique determination of ω, y, j and k follows from Claim 6.4.

The set E as defined in the statement of the Proposition is $K(E_2 \cup uE_2)$. By Claim 6.5, the fact that an element of $GL_2(\mathbb{R})$ factors uniquely as the product of an element of K times an element of T, and $T = T^+ \cup uT^+$, we have that

(i) $Ep^{-1} \bigcap Eq^{-1} = \emptyset$ for all $p, q \in P, p \neq q$,
(ii) $GL_2(\mathbb{R}) = \bigcup_{p\in P} Ep^{-1}$.

This completes the proof that (F, P) forms a tiling system for the orbit O, where $F = \{\chi_b : b \in E\}$. □

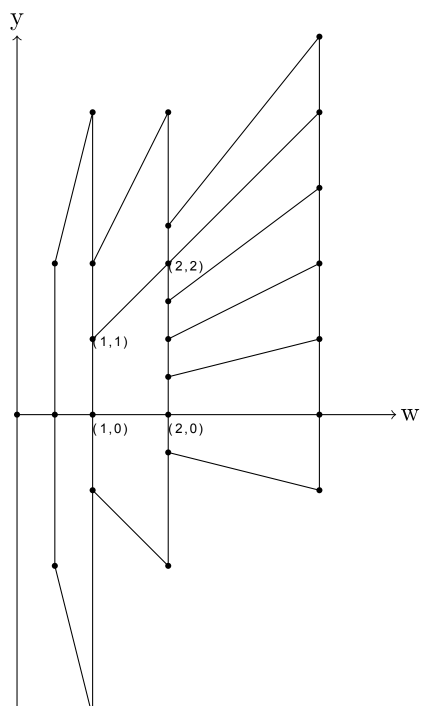

FIGURE 1. Tiles of the form $E_1 p$ sharing a boundary point with E_1

In Figure 1, the trapezoid with vertices $(1,0), (2,0), (1,1)$ and $(2,2)$ in $w - y$ space is E_1 defined in the proof of Proposition 6.3.

Let $P_0 = \{p \in P : \overline{p \cdot F} \cap \overline{F} \neq \emptyset\} = \{p \in P : \overline{Ep^{-1}} \cap \overline{E} \neq \emptyset\}$. Then $\{p \cdot F : p \in P_0\}$ consists of F and the shifts of F by elements of P that are contiguous to F. Figure 1 shows that 11 matrices from P_1 are needed to move E_1 to all adjacent positions (counting the identity matrix). Once the α dimension is added one needs to factor in three intervals of α ($1/2 \leq \alpha < 1, 1 \leq \alpha < 2$ and $2 \leq \alpha < 4$) for each of the tiles in Figure 1 to get 33 members of P_0 in total. Forming $E = K(E_2 \cup uE_2)$ and its adjacent pieces does not add to P_0. For future use, we list P_0 in the following lemma.

Lemma 6.6. *Let*

$$\Gamma = \{(0,0),(0,-1),(0,1),(1,0),(1,1),(-1,-4),(-1,-3),(-1,-2),(-1,-1),(-1,0),(-1,1)\}.$$

Then $P_0 = \left\{ \begin{pmatrix} 2^{l+k} & 2^{l+k}j \\ 0 & 2^{l-k} \end{pmatrix} : l \in \{-1,0,1\}, (k,j) \in \Gamma \right\}.$

Let $D = \cup\{\overline{p \cdot F} : p \in P_0\}$. Let D° denote the interior of D. This compact set D has a somewhat irregular boundary but $F \subseteq D^\circ$. Let

$$M = \text{card}\{p \in P : D^\circ \bigcap p \cdot D^\circ \neq \emptyset\}. \tag{21}$$

It is clear that $M < \infty$.

Lemma 6.7. *For each* $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in M_2(\mathbb{R}), \chi_b \in D$ *implies* $|b_i| \leq 88$, *for* $1 \leq i \leq 4$.

Proof. If $\chi_b \in D$, then

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha w & \alpha y \\ 0 & \pm\alpha w^{-1} \end{pmatrix} \begin{pmatrix} 2^{-l-k} & -2^{-l+k}j \\ 0 & 2^{-l+k} \end{pmatrix},$$

with the ranges of α, y, w, j, k, and l given in Proposition 6.3 and Lemma 6.6. Thus, for example,

$$|b_2| = |\cos\theta\, 2^{-l+k}(-j\alpha w + \alpha y) \mp \sin\theta\, 2^{-l+k}\alpha w^{-1}| \leq 2^{1+1}(4\cdot2\cdot2+2\cdot2) + 2^{1+1}2 = 88.$$

A similar estimate applies to $|b_4|$ while $|b_1|$ and $|b_3|$ are in fact bounded by 16. $\square$

Let $R = \left\{ \chi_b : b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}, |b_i| \leq 88, 1 \leq i \leq 4 \right\}$. Then the Lebesgue volume of R in $\widehat{A}$ is $|R| = 176^4$. Let $L^2(R) = \{\xi \in L^2(\widehat{A}) : 1_R \xi = \xi\}$, where 1_R is the characteristic function of R. That is, $L^2(R)$ is the closed subspace of $L^2(\widehat{A})$ consisting of all the elements supported on R. We can construct an orthonormal basis of $L^2(R)$ by letting $\Lambda = \left\{ \lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} : \lambda_i \in \frac{1}{176}\mathbb{Z}, 1 \leq i \leq 4 \right\}$ and, for each $\lambda \in \Lambda$, defining

$$e_\lambda(\chi) = \begin{cases} 0 & \text{if } \chi \in \widehat{A} \setminus R \\ |R|^{-1/2}\chi(\lambda) & \text{if } \chi \in R. \end{cases} \tag{22}$$

It is straightforward to show that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis of $L^2(R)$. That is, for any $\eta \in L^2(\widehat{A})$ such that $\eta(\chi) = 0$ almost everywhere on $\widehat{A} \setminus R$,

$$\sum_{\lambda \in \Lambda} \left| \int_{\widehat{A}} |R|^{-1/2}\chi(\lambda)\overline{\eta(\chi)}\, d\chi \right|^2 = \sum_{\lambda \in \Lambda} |\langle e_\lambda, \eta \rangle|^2 = \|\eta\|_2^2. \tag{23}$$

The concept of a discrete frame was introduced in [7] and provides the appropriate setting for the discretization of the 4D CWT.

Definition 6.8. *A discrete frame in a Hilbert space $\mathcal{H}$, with frame bounds $0 < C_1 \leq C_2 < \infty$, is a subset $\mathcal{F}$ of $\mathcal{H}$ such that, for all $\eta \in \mathcal{H}$,*

$$C_1\|\eta\|^2 \leq \sum_{\xi \in \mathcal{F}} |\langle \eta, \xi \rangle|^2 \leq C_2\|\eta\|^2.$$

Note that the pair $(P, \overline{F})$ forms a frame generator in the sense of [2]. As a result, Theorem 3 of [2] yields the following.

Proposition 6.9. *Let $g \in L^2(A)$ satisfy $1_F \leq \widehat{g} \leq 1_D$. Then $\{\rho(\lambda, p)^{-1} g : (\lambda, p) \in \Lambda \times P\}$ is a discrete frame in $L^2(A)$ with frame bounds $C_1 = |R|$ and $C_2 = |R|M$.*

Although M is quite large, so the frame bounds are far from tight, we note that a discrete frame as in Proposition 6.9 can be extremely useful in characterizing function spaces via the methods of [8]. In this regard, observe that there exist Schwartz class functions g satisfying the hypothesis of Proposition 6.9.

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