Images of the Continuous Wavelet Transform

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Abstract. A wavelet, in the generalized sense, is a vector in the Hilbert space, $\mathcal{H}_\pi$, of a unitary representation, π, of a locally compact group, G, with the property that the wavelet transform it defines is an isometry of $\mathcal{H}_\pi$ into $L^2(G)$. We study the image of this transform and how that image varies as the wavelet varies. We obtain a version of the Peter-Weyl Theorem for the class of groups for which the regular representation is a direct sum of irreducible representations.

AMS Subject Classification: Primary 42C40; Secondary 43A65
Keywords: Continuous wavelet transform, square integrable representation, Fourier algebra

1. Introduction

As the theory of wavelets emerged, it was recognized early, see [**7**], that the reconstruction formula for the continuous wavelet transform on $\mathbb{R}$ is a direct consequence of the abstract orthogonality relations for a square-integrable representation of a locally compact group [**3**] when applied to the translation and dilation representation of the group of all affine transformations of $\mathbb{R}$. Here, we take the broad point of view, developed in [**6**], that a continuous wavelet theory may be usefully developed whenever one has a unitary representation of some locally compact group and a vector in the Hilbert space of that representation such that an appropriate analog of the classical reconstruction formula exists. The theory of the continuous shearlet transform (see [**12**]) would be an example fitting within our concept of a continuous wavelet transform. The details will be developed in section 3.

Our goal in this paper is to study, in a general setting, the image of a continuous wavelet transform as a subspace of $L^2(G)$ or of the Fourier algebra $A(G)$, where G is the underlying group, and how the images created by different wavelets interrelate. In particular, in Theorem 4.2 we show that the images of a linearly independent pair of wavelets intersect trivially. In Section 5, a version of the Peter-Weyl Theorem is established for [AR]-groups; that is, groups whose regular representation is a direct sum of irreducible representations. We conclude by formulating the concept of a complete K-orthogonal set of wavelets for a square-integrable irreducible representation and exploring that concept for a particular kind of semi-direct product group.

2. General notations and definitions

Let G be a locally compact group equipped with left Haar integral $\int_G \cdot dx$. The modular function Δ of G is a continuous homomorphism of G into $\mathbb{R}^+$, the multiplicative group of positive real numbers and satisfies $\Delta(y) \int_G f(xy)\, dx = \int_G f(x)\, dx$ whenever the integral on the right makes sense. We will also use that $\int_G f(x^{-1})\, dx = \int_G f(x)\Delta(x^{-1})\, dx$.

Let π be a unitary representation of G in a Hilbert space $\mathcal{H}_\pi$. For vectors ξ and η in $\mathcal{H}_\pi$, the continuous function

$$\phi^\pi_{\xi,\eta} : G \to \mathbb{C}, \quad x \mapsto \langle \pi(x)\xi, \eta \rangle$$

is called the *coefficient function* of G associated with the representation π and vectors $\xi, \eta \in \mathcal{H}_\pi$. One can integrate π to create a non-degenerate norm-decreasing $*$-representation of the Banach $*$-algebra $L^1(G)$ in $\mathcal{B}(\mathcal{H}_\pi)$, the Banach algebra of bounded linear operators on $\mathcal{H}_\pi$, via

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx,$$

for every f in $L^1(G)$ and vectors ξ and η in $\mathcal{H}_\pi$. We use the same symbol π to denote the $*$-representation and the associated unitary representation.

For a locally compact group G, the *Fourier-Stieltjes algebra* of G is the set of all the coefficient functions of G, and is denoted by $B(G)$. Clearly $B(G)$ is a subset of $C_b(G)$, the algebra of bounded continuous functions on G. Eymard [**4**] proved that $B(G)$ is actually a subalgebra of $C_b(G)$ and, moreover, it can be identified with the Banach space dual of $C^*(G)$, the group C*-algebra of G. Thus, for $\varphi \in B(G)$,

$$\|\varphi\|_{B(G)} = \sup \left\{ \int_G \varphi(x) f(x)\, dx : f \in L^1(G), \|f\|_* \le 1 \right\},$$

where, for $f \in L^1(G), \|f\|_* = \sup\{\|\pi(f)\| : \pi \text{ is a representation of } G\}$. The Fourier-Stieltjes algebra together with this dual norm turns out to be a Banach algebra. The *Fourier algebra* of G, denoted by $A(G)$, is the closed subalgebra of the Fourier-Stieltjes algebra generated by its compactly supported elements. In the special case of locally compact Abelian groups, one can identify the Fourier and Fourier-Stieltjes algebras with the L^1-algebra and the measure algebra of the dual group respectively. One can refer to [**5**] for a detailed discussion on representation theory of locally compact groups, and [**4**] for the study of Fourier and Fourier-Stieltjes algebras of locally compact groups.

Let π be a continuous unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. Let $A_\pi(G)$ denote the closed subspace of $B(G)$ generated by the coefficient functions of G associated with π, i.e.

$$A_\pi(G) = \overline{\mathrm{Span}_{\mathbb{C}}\{\phi^\pi_{\xi,\eta} : \xi, \eta \in \mathcal{H}_\pi\}}^{\|\cdot\|_{B(G)}}.$$

It is easy to see that $A_\pi(G)$ is a left and right translation-invariant closed subspace of $B(G)$. Conversely, by Theorem (3.17) of [**1**], any closed subspace of $B(G)$ which is left and right translation-invariant, is of the form $A_\pi(G)$ for some continuous unitary representation π. Moreover, the subspace $A_\pi(G)$ can be realized as a quotient of $\mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}_\pi}$, the projective tensor product of $\mathcal{H}_\pi$ and its conjugate $\overline{\mathcal{H}_\pi}$, through the map P from $\mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}}_\pi$ to $A_\pi(G)$ defined as

$$P : \mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}}_\pi \to A_\pi(G), \quad P(\omega)(x) = \langle \omega, \pi(x) \rangle, \ \forall \omega \in \mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}}_\pi, x \in G.$$

Here, we identify $\mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}}_\pi$ with the trace class operators on $\mathcal{H}_\pi$, the predual of $\mathcal{B}(\mathcal{H}_\pi)$.

In the special case where π is irreducible, the above map defines an isometry between $\mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}}_\pi$ and $A_\pi(G)$ (See Theorem 2.2 and Remark 2.6 in [**1**]). We state a consequence of this as a proposition.

PROPOSITION 2.1. Let π be an irreducible representation of a locally compact group G. Let $\xi, \eta \in \mathcal{H}_\pi$. Then $\|\phi^\pi_{\xi,\eta}\|_{A_\pi(G)} = \|\xi\|\|\eta\|$.

PROOF. It is clear that $\phi^\pi_{\xi,\eta} = P(\xi \otimes \overline{\eta})$. Therefore,

$$\|\phi^\pi_{\xi,\eta}\|_{A_\pi(G)} = \|P(\xi \otimes \overline{\eta})\|_{B(G)} = \|\xi \otimes \overline{\eta}\|_{\mathcal{H}_\pi \otimes^\gamma \overline{\mathcal{H}_\pi}} = \|\xi\|\|\eta\|,$$

as required. □

Let λ_G denote the left regular representation of G. The Hilbert space of λ_G is $L^2(G)$ and, for $x \in G$ and $f \in L^2(G)$, $\lambda_G(x)f(y) = f(x^{-1}y)$, for almost all $y \in G$. It was shown in [**4**] that $A_{\lambda_G}(G) = \{\phi^{\lambda_G}_{f,g} : f, g \in L^2(G)\} = A(G)$.

3. Wavelets and square-integrable representations

Let G be a locally compact group, π be a unitary representation of G on a Hilbert space $\mathcal{H}_\pi$ and $\eta \in \mathcal{H}_\pi$ a nonzero vector. Define the linear map $V_\eta : \mathcal{H}_\pi \to C_b(G)$ by, for each $\xi \in \mathcal{H}_\pi$ and $x \in G$,

$$V_\eta(\xi)(x) = \langle \xi, \pi(x)\eta \rangle.$$

If the operator V_η forms an isometry of $\mathcal{H}_\pi$ into $L^2(G)$ (that is, if the range of V_η consists of square-integrable functions and $\|V_\eta(\xi)\|_{L^2(G)} = \|\xi\|$, for all $\xi \in \mathcal{H}_\pi$), then the vector η is called a *wavelet for* π. Thus, as an isometry, V_η preserves inner products. So $\langle V_\eta(\xi), V_\eta(\xi')\rangle = \langle \xi, \xi' \rangle$, for all $\xi, \xi' \in \mathcal{H}_\pi$. Writing out the inner product on the left hand side yields

$$\int_G \langle \xi, \pi(x)\eta \rangle \langle \pi(x)\eta, \xi' \rangle dx = \langle \xi, \xi' \rangle,$$

for all $\xi, \xi' \in \mathcal{H}_\pi$. This leads to the *reconstruction formula*, for any $\xi \in \mathcal{H}_\pi$,

$$\int_G \langle \xi, \pi(x)\eta \rangle \pi(x)\eta dx = \xi, \tag{3.1}$$

weakly in $\mathcal{H}_\pi$. In fact η is a wavelet exactly when (3.1) holds and, in that case, V_η is called a *continuous wavelet transform*.

If the operator V_η is just a nonzero bounded operator from $\mathcal{H}_\pi$ to $L^2(G)$ then the vector η is called an *admissible vector*. The following propositions list some basic and easily demonstrated properties of admissible vectors and wavelets. See [**6**] for a comprehensive introduction to this theory.

PROPOSITION 3.1. Let η be a nonzero admissible vector for a unitary representation π of a locally compact group G.

(i) If π is irreducible, then η is a nonzero multiple of a wavelet for π.
(ii) If π' is a subrepresentation of π and Q is the orthogonal projection from $\mathcal{H}_\pi$ to $\mathcal{H}_{\pi'}$, then $Q\eta$ is either zero or an admissible vector for π'.

PROPOSITION 3.2. Let η be a wavelet for a unitary representation π of a locally compact group G. Then

(i) the vector η is a cyclic vector for π.

(ii) $\lambda_G(x)V_\eta = V_\eta\pi(x)$ for every $x \in G$.
(iii) if π' is a subrepresentation of π then $Q\eta$ is a wavelet for π', where Q is the orthogonal projection from $\mathcal{H}_\pi$ to $\mathcal{H}_{\pi'}$.

Using the above properties, one sees that the operator V_η forms a unitary equivalence of π with a subrepresentation of λ_G whenever η is a wavelet for π. It is not hard to show that this is a sufficient condition when the unitary representation is irreducible. Namely, an irreducible unitary representation of G admits a wavelet if and only if it is unitarily equivalent to a subrepresentation of λ_G. We refer the reader to [**6**] for more details. The study of wavelets for irreducible unitary representations connects naturally to square-integrable representations. An irreducible representation π is called a *square-integrable* representation if it admits a nonzero square-integrable coefficient function $\phi^\pi_{\xi_1,\xi_2}$ for some $\xi_1, \xi_2 \in \mathcal{H}_\pi$. It has been shown in [**6**] that for an irreducible representation π with a nonzero square-integrable coefficient function $\phi^\pi_{\xi_0,\xi_0}$, the operator V_{ξ_0} is a multiple of an isometry, thus ξ_0 is a multiple of a wavelet.

In this article, we will use the following "orthogonality relations" for square-integrable representations shown in [**3**]. Note that Δ denotes the modular function on G.

THEOREM 3.3. [Duflo-Moore [**3**]] *Let π be a square-integrable irreducible representation of a locally compact group G. Then there is a unique densely defined self-adjoint and positive operator K on $\mathcal{H}_\pi$ which satisfies the following conditions.*

(i) *For every $x \in G$, $\pi(x)K\pi(x)^{-1} = \Delta(x)^{-1}K$ (semi-invariant with weight Δ^{-1}).*
(ii) *$\langle \xi, \pi(\cdot)\eta\rangle$ is square integrable if and only if $\eta \in \mathrm{dom}K^{-\frac{1}{2}}$.*
(iii) *Let $\xi, \xi' \in \mathcal{H}_\pi$ and $\eta, \eta' \in \mathrm{dom}K^{-\frac{1}{2}}$. Then*

$$\langle\langle \xi, \pi(\cdot)\eta\rangle, \langle \xi', \pi(\cdot)\eta'\rangle\rangle_{L^2(G)} = \langle \xi, \xi'\rangle\langle K^{-\frac{1}{2}}\eta', K^{-\frac{1}{2}}\eta\rangle.$$

COROLLARY 3.4. *Let π be an irreducible unitary representation, and $\eta \in \mathcal{H}_\pi$. Then, η is a wavelet if and only if $\eta \in \mathrm{dom}K^{-\frac{1}{2}}$ and $\|K^{-\frac{1}{2}}\eta\| = 1$. Moreover, for every $x \in G$, the vector $\sqrt{\Delta(x)}\pi(x)\eta$ is a wavelet as well.*

DEFINITION 3.5. The operator K of Theorem 3.3 is called the *Duflo-Moore operator* of π.

4. Inside $A_{\overline{\pi}}(G)$ for an irreducible π

Throughout this section, let π be a square-integrable irreducible unitary representation of a locally compact group G, and $\eta \in \mathcal{H}_\pi$. Define $\mathcal{A}_\eta := V_\eta(\mathcal{H}_\pi)$. This section concerns decomposing $A_{\overline{\pi}}(G)$ into blocks of the form $\mathcal{A}_\eta$.

LEMMA 4.1. *Let π, η, and $\mathcal{A}_\eta$ be as above. Suppose η is a wavelet for π. Then $\mathcal{A}_\eta$ is a $\|\cdot\|_2$-closed subspace of $L^2(G)$ which is left-invariant. The subspace $\mathcal{A}_\eta$ is not right-invariant if G is non-unimodular. In addition, $\mathcal{A}_\eta$ is a $\|\cdot\|_{B(G)}$-closed subspace of $A_{\overline{\pi}}(G)$, where $\overline{\pi}$ is the representation conjugate to π in the Hilbert space $\overline{\mathcal{H}_\pi}$.*

PROOF. It is clear that $\mathcal{A}_\eta$ is a $\|\cdot\|_2$-closed subspace of $L^2(G)$, since η is a wavelet for π. Also, the subspace $\mathcal{A}_\eta$ is left-invariant by Proposition 3.2. Now suppose that G is non-unimodular, and let y be an element of G such that $\Delta(y) \neq 1$.

Let $\xi \in \mathcal{H}_\pi$ be a nonzero vector. Then for $f = V_\eta(\xi)$ with f_y denoting the right translation of f by y,

$$\|f_y\|_2^2 = \int_G |f(xy)|^2 dx = \int_G |f(x)|^2 \Delta(y^{-1}) dx = \Delta(y^{-1})\|f\|_2^2 = \Delta(y^{-1})\|\xi\|^2,$$

since η is a wavelet. Moreover, using Proposition 2.1,

$$\|f_y\|_{B(G)} = \|V_{\pi(y)\eta}(\xi)\|_{B(G)} = \|\pi(y)\eta\|\|\xi\| = \|\eta\|\|\xi\|.$$

Now assume that f_y is an element of $\mathcal{A}_\eta$, i.e. there exists ξ' in $\mathcal{H}_\pi$ such that $f_y = V_\eta(\xi')$. Then, $\|f_y\|_{B(G)} = \|\eta\|\|\xi'\|$. Hence $\|\xi\| = \|\xi'\|$, and $\|f_y\|_2 \neq \|\xi'\|$. But this is a contradiction with η being a wavelet.

To prove the last statement note that $f = \overline{\phi^\pi_{\eta,\xi}}$. Since π is irreducible, so is $\overline{\pi}$. Therefore, by Proposition 2.1,

$$\|f\|_{B(G)} = \|\overline{\phi^\pi_{\eta,\xi}}\|_{B(G)} = \|\eta \otimes \xi\|_{\overline{\mathcal{H}_\pi} \otimes^\gamma \mathcal{H}_\pi} = \|\eta\|\|\xi\|.$$

On the other hand, $\|f\|_2 = \|V_\eta(\xi)\|_2 = \|\xi\|$. Thus,

$$\|f\|_{B(G)} = \|f\|_2\|\eta\|.$$

That is, the L^2-norm and the Fourier-Sieltjes norm are equivalent on $\mathcal{A}_\eta$. Hence $\mathcal{A}_\eta$ is a $\|\cdot\|_{B(G)}$-closed subspace of $A_{\overline{\pi}}(G)$. □

Observe that for each $x \in G$, the subspace $\mathcal{A}_{\pi(x)\eta} = \{\langle \xi, \pi(\cdot)\pi(x)\eta\rangle : \xi \in \mathcal{H}_\pi\}$ is the right x-translation of $\mathcal{A}_\eta$, and is a $\|\cdot\|_{B(G)}$-closed subspace of $A_{\overline{\pi}}(G)$. Note that the proof of Lemma 4.1 implies that the subspaces $\mathcal{A}_{\pi(x)\eta}$ and $\mathcal{A}_\eta$ intersect trivially whenever $\Delta(x) \neq 1$, if η is a wavelet. The following theorem generalizes this fact, and shows that two subspaces $\mathcal{A}_{\eta_1}$ and $\mathcal{A}_{\eta_2}$, for admissible η_1 and η_2, either coincide or intersect trivially.

Theorem 4.2. *Let π be a square-integrable irreducible unitary representation of a locally compact group G. Let η_1 and η_2 be admissible vectors in $\mathcal{H}_\pi$. Then either $\mathcal{A}_{\eta_1} \cap \mathcal{A}_{\eta_2} = \{0\}$, or $\mathcal{A}_{\eta_1} \cap \mathcal{A}_{\eta_2} = \mathcal{A}_{\eta_1} = \mathcal{A}_{\eta_2}$ and the latter case happens if and only if $\eta_1 = \alpha\eta_2$ for some $\alpha \in \mathbb{C}$. If η_1 and η_2 are both wavelets and $\mathcal{A}_{\eta_1} \cap \mathcal{A}_{\eta_2} \neq \{0\}$, then $\eta_1 = \alpha\eta_2$ for some $\alpha \in \mathbb{T}$*

Proof. Assume that $\mathcal{A}_{\eta_1} \cap \mathcal{A}_{\eta_2} \neq \{0\}$; that is, there exist nonzero vectors ξ and ξ' in $\mathcal{H}_\pi$ such that $0 \neq f(\cdot) = \langle \xi, \pi(\cdot)\eta_1\rangle = \langle \xi', \pi(\cdot)\eta_2\rangle$. Note that

$$\langle \xi, \pi(\cdot)\eta_1\rangle = V_{\eta_1}(\xi) \quad \text{and} \quad \langle \xi', \pi(\cdot)\eta_2\rangle = V_{\eta_2}(\xi').$$

Hence by orthogonality relations stated in Theorem 3.3, we have

$$\begin{aligned}
\|f\|_2^2 &= \langle V_{\eta_1}(\xi), V_{\eta_1}(\xi)\rangle = \|K^{-\frac{1}{2}}\eta_1\|^2\|\xi\|^2, \\
\|f\|_2^2 &= \langle V_{\eta_2}(\xi'), V_{\eta_2}(\xi')\rangle = \|K^{-\frac{1}{2}}\eta_2\|^2\|\xi'\|^2, \\
\|f\|_2^2 &= \langle V_{\eta_1}(\xi), V_{\eta_2}(\xi')\rangle = \langle K^{-\frac{1}{2}}\eta_2, K^{-\frac{1}{2}}\eta_1\rangle\langle \xi, \xi'\rangle.
\end{aligned}$$

Thus,

$$\langle K^{-\frac{1}{2}}\eta_2, K^{-\frac{1}{2}}\eta_1\rangle\langle \xi, \xi'\rangle = \|K^{-\frac{1}{2}}\eta_1\|\|K^{-\frac{1}{2}}\eta_2\|\|\xi\|\|\xi'\|.$$

Since all of the above quantities must be nonzero, by the Cauchy-Schwarz inequality, we have

$$|\langle K^{-\frac{1}{2}}\eta_2, K^{-\frac{1}{2}}\eta_1\rangle| = \|K^{-\frac{1}{2}}\eta_1\|\|K^{-\frac{1}{2}}\eta_2\| \quad \text{and} \quad |\langle \xi, \xi'\rangle| = \|\xi\|\|\xi'\|,$$

which implies that

$$K^{-\frac{1}{2}}\eta_2 = \alpha_1 K^{-\frac{1}{2}}\eta_1 \quad \text{and} \quad \xi' = \alpha_2\xi$$

for scalars α_1 and α_2 in $\mathbb{C}\setminus\{0\}$. Recall that $K^{-\frac{1}{2}}$ is injective, so $\eta_2 = \alpha_1\eta_1$. Clearly, each set $\mathcal{A}_\eta$ forms a vector subspace of $L^2(G)$. Hence

$$\mathcal{A}_{\eta_2} = \mathcal{A}_{\alpha_1\eta_1} = \overline{\alpha_1}\mathcal{A}_{\eta_1} = \mathcal{A}_{\eta_1},$$

which proves the first statements of the theorem.

Moreover, if η_1 and η_2 are wavelets, we have $\|K^{-\frac{1}{2}}\eta_1\| = \|K^{-\frac{1}{2}}\eta_2\| = 1$, which implies that $|\alpha_1| = 1$. □

Corollary 4.3. *Let π be a square-integrable irreducible unitary representation of a locally compact group G and let η be an admissible vector for π. Let $x \in G$ be such that $\mathcal{A}_{\pi(x)\eta} \cap \mathcal{A}_\eta \neq \{0\}$. Then $\pi(x)\eta = \alpha\eta$, for some $\alpha \in \mathbb{T}$.*

Proof. By Theorem 4.2, $\pi(x)\eta = \alpha\eta$ for some $\alpha \in \mathbb{C}$. But $\pi(x)$ is a unitary, so $|\alpha| = 1$. □

Theorem 4.4. *Let π be a square-integrable irreducible unitary representation of a locally compact group G and let η be a wavelet for π. Then $\Sigma_{x\in G}\mathcal{A}_{\pi(x)\eta}$ is $\|\cdot\|_{B(G)}$-dense in $A_{\overline{\pi}}(G)$*

Proof. Observe that $\Sigma_{x\in G}\mathcal{A}_{\pi(x)\eta}$ is a left and right translation invariant subspace of $A_{\overline{\pi}}(G)$. Therefore, $\overline{\Sigma_{x\in G}\mathcal{A}_{\pi(x)\eta}}^{\|\cdot\|_{B(G)}}$ is of the form $A_\sigma(G)$ for a unitary representation σ of G. Since $A_\sigma(G) \subseteq A_{\overline{\pi}}(G)$, the representation σ is a subrepresentation of $\overline{\pi}$ by Corollary (3.14) of [**1**]. This implies that $\Sigma_{x\in G}\mathcal{A}_{\pi(x)\eta}$ is $\|\cdot\|_{B(G)}$-dense in $A_{\overline{\pi}}(G)$, as $\overline{\pi}$ is irreducible and has no proper subrepresentation. □

Example 4.5. Let G be the group of orientation preserving affine transformations of the real line. Then G is the semidirect product $\mathbb{R} \rtimes \mathbb{R}_+$, where $\mathbb{R}_+$ acts on $\mathbb{R}$ by multiplication. In [**11**], it has been shown that

$$A(G) = A_{\pi_+}(G) \oplus_{\ell^1} A_{\pi_-}(G),$$

where $\pi_\pm$ are inequivalent, irreducible unitary representations of G on the Hilbert space $L^2(\mathbb{R}_+^*, dt/t)$ defined by, for $(b,a) \in G$ and $\xi \in L^2(\mathbb{R}_+^*, dt/t)$,

$$\pi_\pm(b,a)\xi(t) := e^{\mp 2\pi ibt}\xi(at),$$

for almost all $t \in \mathbb{R}_+^*$. Consider a continuous compactly supported function η on $\mathbb{R}_+$ which is 1 on $[\frac{1}{2}, 1]$ and nonnegative everywhere else. It is known, see [**10**] for details, that η is a multiple of a wavelet. By normalizing if necessary, we assume that η is a wavelet. Clearly, if $\pi_\pm(b,a)\eta = \alpha\eta$ for some $\alpha \in \mathbb{T}$, then $a = 1$ and $b = 0$. Thus, by Corollary 4.3, $\mathcal{A}_{\pi_\pm(x_1)\eta} \cap \mathcal{A}_{\pi_\pm(x_2)\eta} = \{0\}$ whenever $x_1 \neq x_2$ in G.

Note that π_+ and π_- are unitarily equivalent, respectively, to the two irreducible subrepresentations of the classical wavelet representation ρ acting on $L^2(\mathbb{R})$, where

$$\rho(b,a)f(t) = a^{-1/2}f\left(\frac{t-b}{a}\right),$$

for $t \in \mathbb{R}, (b,a) \in G$, and $f \in L^2(\mathbb{R})$.

In fact, the phenomenon illustrated by Example 4.5 is somewhat general.

THEOREM 4.6. *Let π be a square-integrable irreducible unitary representation of a locally compact group G and suppose that G has no nontrivial compact subgroup. Let η be any wavelet for π. Then $\mathcal{A}_{\pi(x_1)\eta} \cap \mathcal{A}_{\pi(x_2)\eta} = \{0\}$ for any $x_1, x_2 \in G, x_1 \neq x_2$.*

PROOF. Let $x \in G$ be such that $\mathcal{A}_{\pi(x)\eta} \cap \mathcal{A}_\eta \neq \{0\}$. Then, by Theorem 4.2, $\mathcal{A}_{\pi(x^n)\eta} = \mathcal{A}_\eta$ for all $n \in \mathbb{Z}$. Let K denote the closed subgroup of G generated by x. Since π is equivalent to a subrepresentation of the regular representation, $V_\eta \eta$ vanishes at infinity. On the other hand, Corollary 4.3 and continuity implies that

$$|V_\eta \eta(y)| = |\langle \eta, \pi(y)\eta\rangle| = \|\eta\|^2,$$

for all $y \in K$. Therefore K is compact and hence $K = \{e\}$, since G is compact free. If $x_1, x_2 \in G$ satisfy $\mathcal{A}_{\pi(x_1)\eta} \cap \mathcal{A}_{\pi(x_2)\eta} \neq \{0\}$, then $\mathcal{A}_{\pi(x_1)\eta} = \mathcal{A}_{\pi(x_2)\eta}$, which implies $\mathcal{A}_{\pi(x_1 x_2^{-1})\eta} = \mathcal{A}_\eta$. So $x_1 = x_2$. □

5. The $\mathcal{A}_\eta$ as subspaces of $L^2(G)$

We continue with the assumption that π is a square-integrable irreducible representation of a locally compact group G. If η is a nonzero admissible vector for π, then η is a scalar multiple of a wavelet η' and $\mathcal{A}_\eta = \mathcal{A}_{\eta'}$ is a closed subspace of $L^2(G)$. Let $\mathcal{K}_\pi$ denote the smallest closed subspace of $L^2(G)$ that contains $\mathcal{A}_\eta$ for every admissible vector η for π. Fix any wavelet ω for π. Since π is irreducible, $\{\pi(x)\omega : x \in G\}$ is total in $\mathcal{H}_\pi$. Thus $\mathcal{K}_\pi = \overline{\langle \cup \{\mathcal{A}_{\pi(x)\omega} : x \in G\}\rangle}^{L^2(G)}$. Therefore, $\mathcal{K}_\pi$ is a closed subspace of $L^2(G)$ that is invariant under both left and right translations.

If G happens to be compact, any irreducible representation is finite dimensional and square-integrable. In that case, let d_π denote the dimension of $\mathcal{H}_\pi$. From the classical orthogonality relations, one sees that the operator K of Theorem 3.3 is simply $d_\pi I$, where I is the identity operator of $\mathcal{H}_\pi$. Let $\{\nu_1, \cdots, \nu_{d_\pi}\}$ be an orthonormal basis of $\mathcal{H}_\pi$. For $1 \leq j \leq d_\pi$, let $\eta_j = d_\pi^{1/2}\nu_j$. So each η_j is a wavelet for π. Moreover, the orthogonality relations also tell us that $\mathcal{A}_{\eta_j} \perp \mathcal{A}_{\eta_k}$ if $1 \leq j \neq k \leq d_\pi$. Since the linear span of $\{\eta_1, \cdots, \eta_{d_\pi}\}$ is $\mathcal{H}_\pi$,

$$\mathcal{K}_\pi = \oplus_{j=1}^{d_\pi} \mathcal{A}_{\eta_j}.$$

Moreover, $L^2(G) = \oplus_{\pi \in \widehat{G}} \mathcal{K}_\pi$. This is the essential content of the Peter–Weyl Theorem.

With the appropriate interpretation, this generalizes to a class of non-compact groups G, under the assumption of separability.

DEFINITION 5.1. Let G be a locally compact group, π be a square-integrable irreducible representation of G and K the Duflo-Moore operator of π. A collection $\{\eta_j : j \in J\}$ of vectors in $\mathrm{dom} K^{-1/2}$ is called a *complete K-orthogonal set of wavelets* for π if $\{\eta_j : j \in J\}$ is total in $\mathcal{H}_\pi$ and $\{K^{-1/2}\eta_j : j \in J\}$ is orthonormal.

If $\{\eta_j : j \in J\}$ is a complete K-orthogonal set of wavelets for π then, by Theorem 3.3, the $\mathcal{A}_{\eta_j}$ are mutually orthogonal closed subspaces of $\mathcal{K}_\pi$ whose unions span $\mathcal{K}_\pi$. Thus

$$\mathcal{K}_\pi = \oplus_{j \in J} \mathcal{A}_{\eta_j}.$$

THEOREM 5.2. *Let G be a separable locally compact group, π be a square-integrable irreducible representation of G and K the Duflo-Moore operator of π. There exists a countable set $\{\eta_j : j \in J\}$ which is a complete K-orthogonal set of wavelets for π. Moreover, if η is a fixed wavelet, each η_j can be constructed as a finite linear combination of $\{\pi(x)\eta : x \in G\}$.*

PROOF. Fix a wavelet η for π. Let $\{x_j : j \in J'\}$ be a countable dense subset of G. Then $\{\pi(x_j)\eta : j \in J'\}$ is total in $\mathcal{H}_\pi$. Recall that $\mathrm{dom}K^{-1/2}$ consists of exactly the admissible vectors for π and that $\pi(x_j)\eta \in \mathrm{dom}K^{-1/2}$ for each $j \in J'$. Moreover, $K^{-1/2}$ is injective on its domain. Perform the Gram–Schmidt process on the countable set $\{K^{-1/2}\pi(x_j)\eta : j \in J'\}$ and pull the resulting linear combinations back through $K^{-1/2}$ to produce a countable set $\{\eta_j : j \in J\}$ of vectors in $\mathrm{dom}K^{-1/2}$ which is total in $\mathcal{H}_\pi$ and such that $\{K^{-1/2}\eta_j : j \in J\}$ is orthonormal. $\square$

REMARK 5.3. The above theorem appears as Theorem 2.33 in [**6**] without the assumption of separability. However, the sketch of the proof in [**6**] overlooks the fact that Gram–Schmidt requires the initial set of vectors to be countable. That is why we have included the argument here.

A locally compact group G is called an [AR]-group if the left regular representation, λ_G, is the direct sum of irreducible representations (see [**16**] and [**17**]). Let

$$\widehat{G}^r = \{\pi \in \widehat{G} : \pi \text{ is equivalent to a subrepresentation of } \lambda_G\}.$$

We use the symbol π for both an equivalence class of irreducible representations and a particular member of that class. When λ_G is a direct sum of irreducibles, we have left invariant closed subspaces $\mathcal{L}_\pi, \pi \in \widehat{G}^r$, of $L^2(G)$ such that λ_G restricted to $\mathcal{L}_\pi$ is equivalent to a multiple of π, for each $\pi \in \widehat{G}^r$, and $L^2(G) = \oplus_{\pi\in\widehat{G}^r}\mathcal{L}_\pi$. In light of Theorem 5.2, $\mathcal{L}_\pi = \mathcal{K}_\pi$, for each $\pi \in \widehat{G}^r$. This amounts to a Peter–Weyl theory for separable [AR]-groups.

An example will demonstrate the concrete nature of the conditions the η_j appearing in Theorem 5.2 must satisfy.

EXAMPLE 5.4. Fix $c \in \mathbb{R}, c \neq 0$. Let

$$H_c = \left\{ \begin{pmatrix} a & 0 \\ b & a^c \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$$

act on $\mathbb{R}^2$ with the natural matrix action. Form the semidirect product

$$G_c = \mathbb{R}^2 \rtimes H_c = \{[x, h] : x \in \mathbb{R}^2, h \in H_c\},$$

equipped with the group product $[x, h][y, k] = [x + hy, hk]$, for $[x, h], [y, k] \in G$. When $c = 1/2$, this is the shearlet group [**12**]. For general c, this family of groups was investigated in [**15**]. From [**15**], or using elementary Mackey theory [**13**], [**14**], or [**10**], it is easy to show that G_c is an [AR]-group and $\widehat{G_c}^r = \{\rho_+, \rho_-\}$, were ρ_+ can be realized as follows. There is an analogous description of ρ_- with the upper half plane replaced by the lower half plane. The Hilbert space of ρ_+ is $\mathcal{H}_{\rho_+} = \{f \in L^2(\mathbb{R}^2) : \mathrm{supp}\widehat{f} \subseteq \mathcal{O}^+\}$, where $\mathcal{O}^+$ is the upper half plane and

$$\rho_+\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & a^c \end{pmatrix}\right] f\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{a^{c+1}}} f\begin{pmatrix} (y_1 - x_1)/a \\ \frac{y_2 - x_2 - a^{-1}b(y_1 - x_1)}{a^c} \end{pmatrix},$$

for $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$, $\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & a^c \end{pmatrix}\right] \in G_c$, and $f \in \mathcal{H}_{\rho_+}$.

Admissibility conditions for ρ_+ were worked out in [**15**]. This determines the Duflo–Moore operator, K, for this representation. Theorem 5.2 in this setting gives the following method for decomposing the left regular representation of G_c into an infinite multiple of ρ_+ plus an infinite multiple of ρ_-.

Construct $\{\eta_j : j \in J\}$ as a total set in $L^2(\mathcal{O}^+, da\,db)$ such that $\{\eta_j : j \in J\}$ is orthonormal in the weighted L^2-space, $L^2\left(\mathcal{O}^+, \frac{da\,db}{a^c}\right)$. For each $j \in J$, let $w_j \in L^2(\mathbb{R}^2)$ satisfy $\widehat{w_j} = \eta_j$. Then

$$V_{w_j} f[x,h] = \int_{\mathbb{R}^2} f(y)\overline{\rho_+[x,h]w_j(y)}\,dy,$$

for $[x,h] \in G_c, f \in \mathcal{H}_{\rho_+}$. If we let $\mathcal{A}_{w_j} = V_{w_j}\mathcal{H}_{\rho_+}$, then λ_{G_c} restricted to $\mathcal{A}_{w_j}$ is equivalent to ρ_+, for each $j \in J$, and we have $\mathcal{K}_{\rho_+} = \oplus_{j\in J}\mathcal{A}_{w_j}$. Similarly for ρ_- and $L^2(G_c) = \mathcal{K}_{\rho_+} \oplus \mathcal{K}_{\rho_-}$.

We conclude by formulating the construction of Example 5.4 in a more general setting. Let G be a locally compact group of the form $A \rtimes H$, where A is an abelian locally compact group and H is a σ-compact locally compact group acting on A via $(h,a) \to h \cdot a$, for $h \in H, a \in A$. Then H acts on the dual group $\widehat{A}$ by, for $h \in H, \chi \in \widehat{A}, (h\cdot\chi)(a) = \chi(h^{-1}\cdot a)$, for all $a \in A$. Further, assume that there exists an open free H-orbit $\mathcal{O}$ in $\widehat{A}$. Then for fixed $\omega \in \mathcal{O}$, the map $h \to h^{-1}\cdot\omega$ is a homeomorphism of H onto $\mathcal{O}$. See [**9**] and Sections 7.2 and 7.3 of [**10**] for a treatment of this situation.

Let δ denote the homomorphism of H into $\mathbb{R}_+^*$ such that, for any integrable function g on A and any $h \in H$, we have $\delta(h)\int_A g(h\cdot a)\,da = \int_A g(a)\,da$. There is a square-integrable irreducible representation $\pi_{\mathcal{O}}$ of G associated with $\mathcal{O}$ which can be realized as follows. The Hilbert space of $\pi_{\mathcal{O}}$ is $L^2(\mathcal{O}, m)$, where the measure m on $\mathcal{O}$ is the restriction of the Haar measure of $\widehat{A}$, and, for $(a,h) \in G, \xi \in L^2(\mathcal{O}, m)$,

$$\pi_{\mathcal{O}}(a,h)\xi(\chi) = \delta(h)^{1/2}\chi(a)\xi(h^{-1}\cdot\chi),$$

for all $\chi \in \mathcal{O}$ (Proposition 7.17, [**10**]).

Thus, there are two relevant measures on the orbit $\mathcal{O}$, the Haar measure of $\widehat{A}$ restricted to $\mathcal{O}$ and the left Haar measure on H, moved to $\mathcal{O}$ via the homeomorphism $h \to h^{-1}\cdot\omega$. Let μ denote the latter measure. Then

$$\int_{\mathcal{O}} \varphi(\chi)\,d\mu(\chi) = \int_H \varphi(h^{-1}\cdot\omega)\,dh, \tag{5.1}$$

for any $\varphi \in C_c(\mathcal{O})$. Note that the right hand side of (5.1) is independent of the choice of $\omega \in \mathcal{O}$. We use $L^2(\mathcal{O}, \mu)$ to denote the L^2-space when μ is the measure on $\mathcal{O}$.

The computation in the proof of Theorem 7.19 of [**10**], after adjusting the notation, shows that, for $\xi, \eta \in L^2(\mathcal{O}, m)$,

$$\|V_\eta\xi\|_2^2 = \int_{\mathcal{O}} |\xi(\chi)|^2 d\chi \int_{\mathcal{O}} |\eta(\chi)|^2 d\mu(\chi). \tag{5.2}$$

This implies that $\|K^{-1/2}\eta\|^2 = \int_{\mathcal{O}} |\eta(\chi)|^2 d\mu(\chi)$ and, via polarization, that

$$\langle K^{-1/2}\eta_1, K^{-1/2}\eta_2\rangle = \int_{\mathcal{O}} \eta_1(\chi)\overline{\eta_2}(\chi)d\mu(\chi) = \int_H \eta_1(h^{-1}\cdot\omega)\overline{\eta_2}(h^{-1}\cdot\omega)dh, \tag{5.3}$$

for all admissible η_1 and η_2 in $L^2(\mathcal{O}, m)$. Note that the set of admissible vectors in $L^2(\mathcal{O}, m)$ is exactly $L^2(\mathcal{O}, m) \cap L^2(\mathcal{O}, \mu)$ and this intersection makes sense because the two measures in question are mutually absolutely continuous. Observe that a complete K-orthogonal set of wavelets for $\pi_{\mathcal{O}}$ is a collection $\{\eta_j : j \in J\}$ of functions in $L^2(\mathcal{O}, m) \cap L^2(\mathcal{O}, \mu)$ which is total in $L^2(\mathcal{O}, m)$ and orthonormal in $L^2(\mathcal{O}, \mu)$.

Remark 5.5. In all the examples known to the authors where a σ-compact locally compact group H acts on an abelian locally compact group A in such a manner that there exists an open free H-orbit in $\widehat{A}$, the union of all of the open free H-orbits is co-null in $\widehat{A}$ and, as a result, $A \rtimes H$ is an [AR]-group.

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