MULTIRESOLUTION ANALYSIS AND HARR-LIKE WAVELET BASES ON LOCALLY COMPACT GROUPS

Qingde Yang

Department of Mathematics and Statistics, University of Saskatchewan

Saskatoon, SK S7N 5E6 Canada, yangq@math.usask.ca

Keith F. Taylor

Department of Mathematics and Statistics, Dalhousie University

Halifax, NS B3H 3J5 Canada, keith.taylor@mathstat.dal.ca

Abstract. The multiresolution analysis (MRA) on certain non-abelian locally compact groups G is considered. Characterizations for a refinable function to generate an MRA in $L^2(G)$ are given. Here, no regularity properties or decay conditions are placed on the scaling functions. MRAs for $L^2(G)$ generated by a self-similar tile as a scaling function are shown and Haar-like wavelet bases are constructed. Concrete examples related to Heisenberg group are provided to illustrate the theorems.

Keywords: multiresolution analysis, non-abelian locally compact group, scaling function, Haar-like wavelet base, Heisenberg group.

1. INTRODUCTION

The theory of wavelets in the Hilbert space $L^2(\mathbb{R}^d)$ has been studied extensively in recent decades. The principal framework for constructing and understanding wavelet bases for the Hilbert space $L^2(\mathbb{R}^d)$ is the concept of multiresolution analysis (MRA) [**4, 8, 10**]. For a general Hilbert space $\mathcal{H}$, the notion of MRA can be formulated with respect to a distinguished affine structure (Π, σ) in the following way [**1**]. Let Π be a countable, discrete subgroup of the group of unitary operators on $\mathcal{H}$ and σ be another unitary operator on $\mathcal{H}$ satisfying $\sigma^{-1}\Pi\sigma\subseteq\Pi$ and $1 < [\Pi : \sigma^{-1}\Pi\sigma] < \infty$. An MRA with the scaling vector (or function) $\phi\in\mathcal{H}$ for the affine structure (Π, σ) is a doubly infinite sequence $\{V_j : j\in\mathbb{Z}\}$ of closed subspaces of $\mathcal{H}$ with the following properties:

(i) $\{u\phi : u\in\Pi\}$ is an orthonormal basis for V_0;

(ii) $V_j = \sigma^j V_0$, for all $j\in\mathbb{Z}$;

(iii) $V_j\subseteq V_{j+1}$, for all $j\in\mathbb{Z}$;

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (the triviality of the intersection);

(v) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = \mathcal{H}$ (the density of the union).

A scaling vector $\phi\in\mathcal{H}$ is called *refinable* if $\phi\in\overline{< \{\sigma(u\phi) : u\in\Pi\} >}$, the closure of the finite linear combinations of the functions from $< \{\sigma(u\phi) : u\in\Pi\} >$. If ϕ is refinable, condition (iii) in the above definition is satisfied. Using the unitary operator σ and the space V_0, we get a sequence $\{V_j : j\in\mathbb{Z}\}$ of nested closed subspaces of $\mathcal{H}$. In order to construct an MRA, the triviality of the intersection and the density of the union become two crucial conditions. We will see in section 3 that the triviality of the intersection is a direct consequence of conditions (i), (ii), and (iii). The question now is: when does the density of the union hold? To answer this question, let us first take a look at the Hilbert space $L^2(\mathbb{R}^d)$.

In $L^2(\mathbb{R}^d)$, the above mentioned affine structure is $\Pi = \{T_k : k\in\mathbb{Z}^d\}$ and $\sigma = \sigma_D$, where T_x is the translation operator defined by $T_x f(\cdot) := f(\cdot - x)$ for any $f\in L^2(\mathbb{R}^d)$ and σ_D is the dilation operator defined by $\sigma_D f(\cdot) = \delta_D^{1/2} f(D\cdot)$ for any $f\in L^2(\mathbb{R}^d)$ with D being the dilation matrix and $\delta_D = |\det(D)|$. A special dilation matrix is $D = 2I$, where I is the identity matrix. Suppose $\phi\in L^2(\mathbb{R}^d)$. Define $V(\phi)$ as $\overline{\{T_k\phi : k\in\mathbb{Z}^d\}}$. Then $V(\phi)$ is the smallest closed shift invariant subspace generated by ϕ, that is, $T_k f\in V(\phi)$ for any $f\in V(\phi)$ and any $k\in\mathbb{Z}^d$. Now define $V_j = {\sigma_{2I}}^j V(\phi)$ for any $j\in\mathbb{Z}$. The sequence of the closed subspaces $\{V_j : j\in\mathbb{Z}\}$ is nested if ϕ is refinable. A scaling function must be a refinable function. But the other way around is not true. Boor, DeVore and Ron [**2**] showed that the refinability of ϕ alone is not enough for ϕ to generate an MRA. For a refinable function to generate an MRA, additional conditions are required. They proved that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$ if and only if ϕ is refinable and $\bigcup_{j\in\mathbb{Z}}\text{supp}(\widehat{\phi_j}) = \mathbb{R}^d$ modulo a null-set, where $\phi_j(\cdot) := {\sigma_{2I}}^j\phi(\cdot) = 2^{dj/2}\phi(2^j\cdot)$, $\widehat{\phi_j}$ is the Fourier transform of ϕ_j, and $\text{supp}(\widehat{\phi_j}) := \{\xi\in\mathbb{R}^d \ : \ \widehat{\phi_j}(\xi)\neq 0\ \}$. They also gave a sufficient condition for the density of the union: $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^d)$ if ϕ is refinable and $\widehat{\phi}$ is nonzero a.e. in some neighborhood of the origin. This sufficient condition can be easily proven. If $\widehat{\phi}$ is nonzero a.e. in some neighborhood of the origin, then we see that $\bigcup_{j\in\mathbb{Z}}\text{supp}(\widehat{\phi_j}) = \mathbb{R}^d$ because $\widehat{\phi_j}(\cdot) = \widehat{\phi}(\cdot/2^j)$.

In this paper, we are interested in MRA defined over a more abstract Hilbert space. More specifically, we are interested in MRA defined over Hilbert spaces of the form $L^2(G)$, where G is some locally compact group. In the case that G is an abelian locally compact group, Dahlke has successfully extended the concept of MRA to $L^2(G)$ [**3**]. The main purpose of this paper is to develop some characterizations of functions which can serve as scaling functions in the more general case where the Hilbert space is $L^2(G)$, where G is a locally compact group, but may or may not be abelian. From the abstract harmonic analysis point of view, [**2**] uses the information on the Plancherel side to describe the qualities of a refinable function that can generate an MRA for the space $L^2(\mathbb{R}^d)$. For a general locally compact group G, it may be impossible to determine the Plancherel measure on the dual space $\widehat{G}$. Thus, the information on the Plancherel side is not available in general in this case. In contrast to [**2**], we only use the concepts coming within the space $L^2(G)$ to develop characterizations of functions that can serve as scaling functions without looking at the Plancherel side. Here, we note that we do not need to assume that the scaling functions have regularity properties, nor impose any decay conditions. To make the argument more general, we only assume that the scaling functions are elements in the space $L^2(G)$.

Analogous to the construction of MRA in the space $L^2(\mathbb{R}^d)$, to build an MRA on a general group G, the group G must have a uniform lattice subgroup Γ and a dilative automorphism α such that $\alpha(\Gamma)\subseteq\Gamma$ and $1 < [\Gamma : \alpha(\Gamma)] < \infty$ (see section 2 for the precise definition). We call (Γ, α) a *scaling system.* The corresponding affine structure on $L^2(G)$ is then provided by (Π, σ_α), where $\Pi = \lambda_G(\Gamma)$, λ_G is the left regular representation of G, σ_α is the unitary operator defined by $\sigma_\alpha f(x) = \delta_\alpha^{1/2} f(\alpha(x))$ and δ_α is the factor by which α scales the Haar measure on G.

An important example is $G = \mathbb{H}$, the three dimensional nilpotent Lie group. $\mathbb{H}$ is realized as follows: $\mathbb{H} = \{(p,q,t) : p,q,t \in \mathbb{R}\}$ with group product given by $(p,q,t)(p',q',t') = (p+p', q+q', t+t' + \frac{1}{2}(pq' - qp'))$. A choice for a uniform lattice in $\mathbb{H}$ is $\Gamma = \{(m,n,l/2) : m,n,l \in \mathbb{Z}\}$ and a compatible dilative automorphism is $\alpha(p,q,t) = (2p, 2q, 2^2 t)$, for all $(p,q,t) \in \mathbb{H}$. This example and variations on it will be used to illustrate our main results later on.

For any locally compact group G and a subset $F \subseteq L^2(G)$, we say that the family F is a *left zero divisor* in $L^2(G)$ if there exists a $g \in L^2(G)$, $g \neq 0$, such that $f * g = 0$, for all $f \in F$. If α is an automorphism of G, a single function $f \in L^2(G)$ is called *α-substantial* if $\{\sigma_\alpha^j f(\cdot) : j \in \mathbb{Z}\}$ is not a left zero divisor in $L^2(G)$.

If $G = \mathbb{R}$ and $F \subseteq L^2(G)$, then F is a left zero divisor in $L^2(G)$ if and only if there exists a measurable subset E of $\mathbb{R}$, of positive Lebesque measure, such that $\widehat{f}(\omega) = 0$, for all $f \in F$ and almost all $\omega \in E$. If α is the automorphism given by $\alpha(t) = 2t$, for all $t \in \mathbb{R}$, then a function $f \in L^2(\mathbb{R})$ is α-substantial if and only if there exists a measurable subset $A \subseteq \mathbb{R}$ such that $\widehat{f}(\omega) \neq 0$, for almost all $\omega \in A$ and $\bigcup_{j\in\mathbb{Z}} 2^j A = \mathbb{R}$. In the particular case that $\widehat{f}(\omega) \neq 0$ for almost all ω in a neighborhood of the origin, then $\bigcup_{j\in\mathbb{Z}} 2^j \mathrm{supp}(\widehat{f}) = \mathbb{R}$. So any such function f is α-substantial.

Despite the lack of tools from Fourier analysis to help us to recognize α-substantial functions in the case of a general locally compact group G, we are able to show that if α is dilative, $f \in L^2(G)$, $f \geq 0$, $f \neq 0$ and is of compact support, then f is α-substantial. This is done in section 3.

Consider a locally compact group G. A subspace X of $L^2(G)$ is called *left shift invariant* if $\lambda_G(\gamma) X \subseteq X$, for all $\gamma \in \Gamma$. For $\phi \in L^2(G)$, let $V(\phi)$ denote the smallest left shift invariant closed subspace of $L^2(G)$ containing ϕ. Define $V_j = \sigma_\alpha^j\, V(\phi)$, for all $j \in \mathbb{Z}$. Then the fact that ϕ is refinable implies that $V_0 \subseteq V_1$ (then $V_j \subseteq V_{j+1}$, for all $j \in \mathbb{Z}$). Define $\phi_j(x) = \sigma_\alpha^j \phi(x) = \delta_\alpha^{j/2} \phi(\alpha^j(x))$, for any $x \in G$, $j \in \mathbb{Z}$.

The main results of this paper are summarized in the following theorems.

Theorem 3.1(Triviality of the intersection): *Let G be a locally compact group with a scaling system (Γ, α). Let ϕ be a refinable function of $L^2(G)$ and V_j, $j\in\mathbb{Z}$ defined as above. Suppose that the shifts of ϕ, that is, $\{L_\gamma \phi \,:\, \gamma\in\Gamma\}$, constitute an orthonormal basis for V_0, then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.*

Theorem 3.5 (Density of the union): *Let ϕ be a refinable function in $L^2(G)$ and V_j, $j\in\mathbb{Z}$ defined as above. Then the following are equivalent:*

(a) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$

(b) $\{\phi_j\}_{j\in\mathbb{Z}}$ *is a left nonzero divisor in* $L^2(G)$

(c) ϕ *is* $\alpha-$*substantial.*

Theorem 4.3: *Let G be a locally compact group and (Γ, α) a scaling system on G. Suppose that there exists a self-similar tile T for (Γ, α) on G. Then $\phi = \chi_T$ will generate an MRA for the space $L^2(G)$.*

Theorem 4.6: *The MRA generated by a self-similar tile will always guarantee a Haar-like orthonormal wavelet basis for the space $L^2(G)$.*

The rest of the paper is arranged as follows. Section 2 provides the basic concepts, definitions of the terms and some basic results on a scaling system and its corresponding affine structure. In section 3, we first prove the intersection triviality theorem. Then we prove several propositions that lead to the proof of theorem 3.5 as stated above. Section 4 concerns with refinable functions of self-similar tile in the space $L^2(G)$, MRAs generated by using these refinable functions as scaling functions, and Haar-like wavelet bases associated with these MRAs. In section 5, we turn to the examples of the Heisenberg group. We draw upon the idea of [**12**] to give an explicit construction of refinable function on the Heisenberg group. This construction will provide examples to illustrate our theorems established in sections 3 and 4.

2. BASIC CONCEPTS

Let G be a locally compact group with left Haar measure m_G. Integration with respect to m_G will be denoted simply by $\int_G f(x)\ dx$, for any appropriate complex-value function f on G. Let $L^2(G)$ denote the Hilbert space of (equivalence classes of) square integrable complex-valued functions on G with inner product: $< f, g >= \int_G f(x)\overline{g(x)}\ dx$, for $f, g \in L^2(G)$. The left regular representation of G is the faithful homomorphism λ_G of G into the unitary group on $L^2(G)$ given by $\lambda_G(x)f(y) = f(x^{-1}y)$, $\forall x, y \in G$, $f \in L^2(G)$. If the group of unitary operators on $L^2(G)$ is endowed with the strong operator topology, then λ_G is continuous. If α is an automorphism (topological homomorphism and algebraic automorphism) of G, then $f \rightarrow \int_G f(\alpha(x))\ dx$ is a left invariant integral so there exists a positive constant δ_α so that $\int_G f(x)\ dx = \delta_\alpha \int_G f(\alpha(x))\ dx$, for any appropriate function f on G. This means that α induces a unitary operator σ_α on $L^2(G)$ by $\sigma_\alpha f(x) = \delta_\alpha^{1/2} f(\alpha(x))$, for all $x \in G$, $f \in L^2(G)$. Note that, for any $j \in \mathbb{Z}$, α^j is an automorphism of G and

$$\sigma_\alpha{}^j f(x) = \delta_\alpha^{j/2} f(\alpha^j(x)), \text{ for all } x \in G,\ f \in L^2(G).$$

An automorphism α of G is called *dilative* if, for any compact $K \subseteq G$ and any neighborhood U of the identity e of G, there exists an $n_0 \in \mathbb{N}$ such that $K \subseteq \alpha^n(U)$, for all $n \geq n_0$. Note that α is dilative implies that $\delta_\alpha > 1$ but the converse is not true.

A subgroup Γ of G is called a uniform lattice in G if Γ is discrete, countable and G/Γ is compact. Suppose Γ is a uniform lattice in G and α is a dilative automorphism of G such that $\alpha(\Gamma) \subseteq \Gamma$ and $1 < [\Gamma : \alpha(\Gamma)] < \infty$. Then we will call (Γ, α) a scaling system (based on G).

For many of the results of this paper, we assume that (Γ, α) is a scaling system. This condition imposes strong restrictions on the group G and the discrete subgroup Γ. We will not fully investigate these restrictions here, but we need to make a few observations.

Proposition 2.1. *Let (Γ, α) be a scaling system. Then the following conditions hold.*

(a) *G is a unimodular group;*

(b) *Γ is not an open subgroup of G;*

(c) *$m_G(\Gamma) = 0$;*

(d) *For any* $j_0 \in \mathbb{Z}$, $\bigcup_{j \geq j_0} \alpha^{-j}(\Gamma)$ *is dense in* G.

Proof. (a) follows from numbers 1.8 to 1.11 on page 21 in [**11**] for example. To see (b), suppose Γ were an open subgroup of G. Then it is a neighborhood containing the e. Since $\alpha(\Gamma) \subseteq \Gamma$, if $n \geq 1$, then $\alpha^n(\Gamma) \subseteq \Gamma$. Since $1 < [\Gamma : \alpha(\Gamma)]$ and α is an automorphism of G, Γ is not all of G. Taking $U = \Gamma$ and $K = \{x\}$ for some $x \in G \backslash \Gamma$, then we would have that $K = \{x\} \subseteq \alpha^n(\Gamma) \subseteq \Gamma$ by the dilative property of α, which is a contradiction to the fact that $x \in G \backslash \Gamma$. Then (c) follows from (b).

For (d), choose a compact subset K of G such that $G = \bigcup_{\gamma \in \Gamma} K\gamma$; this is possible because Γ is a uniform lattice in G. For any $x \in G$ and any neighborhood W of x, let U be a symmetric ($U^{-1} = U$) neighborhood of e such that $Ux \subseteq W$. Since α is dilative, there exists $n_0 \in \mathbb{N}$ such that $j \geq n_0$ implies $K \subseteq \alpha^j(U)$; that is, $\alpha^{-j}(K) \subseteq U$. Then

$$G = \alpha^{-j}(G) = \bigcup_{\gamma \in \Gamma} \alpha^{-j}(K\gamma) = \bigcup_{\gamma \in \Gamma} \alpha^{-j}(K)\alpha^{-j}(\gamma) \subseteq \bigcup_{\gamma \in \Gamma} U\alpha^{-j}(\gamma).$$

Thus, there exists a $\gamma \in \Gamma$ such that $x \in U\alpha^{-j}(\gamma)$. Hence, $x = u\alpha^{-j}(\gamma)$, for some $u \in U$ and $\alpha^{-j}(\gamma) = u^{-1}x \in Ux \subseteq W$. Therefore, $W \bigcap \alpha^{-j}(\Gamma) \neq \emptyset$, for any $j \geq n_0$. Because $\alpha^{-j}(\Gamma) \subseteq \alpha^{-(j+1)}(\Gamma)$ for all j, $\bigcup_{j \geq j_0} \alpha^{-j}(\Gamma)$ is dense in G for any fixed $j_0 \in \mathbb{Z}$. $\square$

If (Γ, α) is a scaling system and $\phi \in L^2(G)$, we use ϕ to generate a family of closed subspaces of $L^2(G)$ in analogy with the role played by a scaling vector in an MRA. Let $V(\phi)$ denote the closed linear span of $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$. For each $j \in \mathbb{Z}$, let $V_j = {\sigma_\alpha}^j V(\phi)$. A function of the form $\lambda_G(\gamma)\phi$ is called a shift of ϕ, so the shifts of ϕ forms an orthonormal basis exactly when $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$ forms an orthonormal basis of $V(\phi)$. Moreover, the fact that ϕ is refinable exactly means $V_0 \subseteq V_1$. So $V_j \subseteq V_{j+1}$, for all $j \in \mathbb{Z}$.

Proposition 2.2. *Let* (Γ, α) *be a scaling system and* $\phi \in L^2(G)$. *Then the following conditions hold.*

(a) ${\sigma_\alpha}^j \lambda_G(\gamma) {\sigma_\alpha}^{-j} = \lambda_G(\alpha^{-j}(\gamma))$, *for all* $j \in \mathbb{Z}$, $\gamma \in \Gamma$,

(b) *If* $\Pi = \lambda_G(\Gamma)$, *then* (Π, σ_α) *is an affine structure on* $L^2(G)$,

(c) $V_j = \overline{< \{\lambda_G(\nu){\sigma_\alpha}^j \phi : \nu \in \alpha^{-j}(\Gamma)\} >}$, *for* $j \in \mathbb{Z}$.

PROOF. For $f\in L^2(G)$ and $x\in G$, compute

$$\sigma_\alpha{}^j\lambda_G(\gamma)\sigma_\alpha{}^{-j}f(x)=\delta_\alpha^{j/2}\lambda_G(\gamma)\sigma_\alpha{}^{-j}f(\alpha^j(x))=\delta_\alpha^{j/2}\sigma_\alpha{}^{-j}f(\gamma^{-1}\alpha^j(x))$$

$$=f(\alpha^{-j}(\gamma^{-1})x)=\lambda_G(\alpha^{-j}(\gamma))f(x).$$

This establishes (a). In particular, $\sigma_\alpha{}^{-1}\lambda_G(\gamma)\sigma_\alpha=\lambda_G(\alpha(\gamma))$. So, if $\Pi=\lambda_G(\Gamma)$, then $\sigma_\alpha{}^{-1}\Pi\sigma_\alpha=\lambda_G(\alpha(\Gamma))$. Since λ_G is a faithful homomorphism, $[\Pi:\sigma_\alpha{}^{-1}\Pi\sigma_\alpha]=[\Gamma:\alpha(\Gamma)]$ and (Π,σ_α) is an affine structure in the sense of [**1**]. So (b) holds and (c) also follows from (a) and the definition of V_j. □

3. THE CHARACTERIZATIONS OF A SCALING FUNCTION

In this section, the first two theorems stated in the introduction are proven. We first prove the triviality of the intersection because it is the direct consequence of refinability and orthogonal shifts. Then we establish the density of the union by considering several propositions. Finally, we show that the space $L^2(G)$ provides an abundant supply of α-substantial functions.

THEOREM 3.1. *(Triviality of the intersection)* Let G be a locally compact group with scaling system (Γ,α). Let ϕ be a refinable function of $L^2(G)$ and define $V_j=\sigma_\alpha^jV(\phi)$ for $j\in\mathbb{Z}$. Suppose that the shifts of ϕ, that is, $\{\Gamma_\gamma\phi:\gamma\in\Gamma\}$, constitutes an orthogonal basis for V_0, then $\bigcap_{j\in\mathbb{Z}}V_j=\{0\}$.

PROOF. Since ϕ is refinable, $V_j\subseteq V_{j+1}$, for all $j\in\mathbb{Z}$, so

$$\bigcap_{j\in\mathbb{Z}}V_j=\bigcap_{n\in\mathbb{N}}V_{-n}.$$

Since ϕ is a unit vector with orthogonal shifts and σ is unitary, for each $j\in\mathbb{Z}$, $\{\sigma^j(\lambda_G(\gamma)\phi):\gamma\in\Gamma\}$ is a orthonormal basis for V_j. The orthogonal projection P_j of $L^2(G)$ onto V_j is given by

$$P_jf=\sum_{\gamma\in\Gamma}<f,\sigma^j(\lambda_G(\gamma)\phi)>\sigma^j(\lambda_G(\gamma)\phi),\ \forall f\in L^2(G).$$

Let $f\in\bigcap_{j\in\mathbb{Z}}V_j$; so $P_jf=f$, for all $j\in\mathbb{Z}$. Let $\epsilon>0$ be arbitrary. Select a continuous function of compact support, f_1, so that $\|f-f_1\|_2<\epsilon$. Then $\|f-P_jf_1\|_2=\|P_j(f-f_1)\|_2<\epsilon$ and so $\|f\|_2\leq\|P_jf_1\|_2+\epsilon$, for any $j\in\mathbb{Z}$.

Let K be a compact subset of G so that $\mathrm{supp}(f_1)\subseteq K$ and let $M = \sup\{|f_1(x)| : x\in K\}$. Let W be a neighborhood of e in G such that $W\bigcap\Gamma = \{e\}$. Let U be another neighborhood of e such that $UU^{-1}\subseteq W$. Then, for any γ_1, $\gamma_2\in\Gamma$, $\gamma_1\neq\gamma_2$ implies $\gamma_1^{-1}U\bigcap\gamma_2^{-1}U = \emptyset$.

Since α is dilative, there exists $n_0\in\mathbb{N}$ such that $K\subseteq\alpha^n(U)$ for any $n\geq n_0$; that is $\alpha^{-n}(K)\subseteq U$ if $n\geq n_0$. Therefore, for γ_1, $\gamma_2\in\Gamma$, with $\gamma_1\neq\gamma_2$ and $n\geq n_0$, we have $\gamma_1^{-1}\alpha^{-n}(K)\bigcap\gamma_2^{-1}\alpha^{-n}(K) = \emptyset$. Let $E_{-n} = \bigcup_{\gamma\in\Gamma}\gamma^{-1}\alpha^{-n}(K)$. If $n\geq n_0$, we have additivity of the characteristic functions,

$$\chi_{E_{-n}} = \sum_{\gamma\in\Gamma}\chi_{\gamma^{-1}\alpha^{-n}(K)}.$$

Now, fix a point $x\in G\backslash\Gamma$. Since Γ is discrete in the relative topology of G, it is a closed subgroup. Thus, there exists a symmetric neighborhood V of e such that $xV\bigcap\Gamma = \emptyset$. Then x is not in $\gamma^{-1}V$, for any $\gamma\in\Gamma$. Using the dilative nature of α again, there exists $n_1\geq n_0$ such that $n\geq n_1$ implies $\alpha^{-n}(K)\subseteq V$. So, for any $n\geq n_1$, $\chi_{E_{-n}}(x) = 0$. Therefore, the sequence $(\chi_{E_{-n}})_{n=1}^{\infty}$ converges to 0 pointwise on $G\backslash\Gamma$, so it converges to 0 pointwise almost everywhere on G, since $m_G(\Gamma) = 0$. For $n\geq n_1$,

$$\begin{aligned}
\|P_{-n}f_1\|_2^2 &= \sum_{\gamma\in\Gamma} |<f_1, \sigma^{-n}(\lambda_G(\gamma)\phi)>|^2\\
&= \sum_{\gamma\in\Gamma}\delta_\alpha^{-n}|\int_G f_1(x)\overline{\phi(\gamma^{-1}\alpha^{-n}(x))}\,dx|^2\\
&\leq \sum_{\gamma\in\Gamma}\delta_\alpha^{-n}(\int_G |f_1(x)||\phi(\gamma^{-1}\alpha^{-n}(x))|\,dx)^2\\
&\leq M^2\sum_{\gamma\in\Gamma}\delta_\alpha^{-n}(\int_G |\chi_K(x)||\phi(\gamma^{-1}\alpha^{-n}(x))|\,dx)^2\\
&\leq M^2 m_G(K)^2\sum_{\gamma\in\Gamma}\delta_\alpha^{-n}\int_K |\phi(\gamma^{-1}\alpha^{-n}(x))|^2\,dx\\
&= M^2 m_G(K)^2\sum_{\gamma\in\Gamma}\int_{\gamma^{-1}\alpha^{-n}(K)}|\phi(y)|^2\,dy\\
&= M^2 m_G(K)^2\int_G \chi_{E_{-n}}(x)|\phi(y)|^2\,dy\longrightarrow 0, \text{ as } n\rightarrow\infty,
\end{aligned}$$

by the Lebesque Dominated Convergence Theorem. The last inequality in the above calculation is an application of the Cauch-Schartz inequality. Since $\|f\|_2\leq\|P_{-n}f_1\|_2+$

ϵ, for all n, $\|f\|_2 \leq \epsilon$ and $\epsilon > 0$ being arbitrary, we conclude that $f = 0$. Therefore, $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. $\square$

REMARK 3.2. In Theorem 3.1, one can replace the assumption of orthogonal shifts with the assumption that the shifts of ϕ constitute a frame in V_0, as for the $L^2(\mathbb{R}^d)$ situation, but we preferred to write out the clearer argument with the stronger assumption since we need orthogonal shifts elsewhere.

A subspace X of $L^2(G)$ is called *left translation invariant* if $\lambda_G(x)f \in X$, for all $f \in X$ and $x \in G$. For a family of functions $F \subseteq L^2(G)$. Let $X(F)$ denote the smallest closed left translation invariant subspace of $L^2(G)$ which contains F. Obviously,

$$X(F) = \overline{<\{\lambda_G(x)f : x\in G, f\in F\}>} = \{\lambda_G(x)f : x\in G, f\in F\}^{\perp\perp}.$$

Recall that we call F a left zero divisor in $L^2(G)$ if there exists a nonzero g in $L^2(G)$ such that $f*g = 0$, for all $f \in F$.

PROPOSITION 3.3. *Let G be a unimodular locally compact group and let $F \subseteq L^2(G)$. Then $X(F) = L^2(G)$ if and only if F is not a left zero divisor in $L^2(G)$.*

PROOF. For $g \in L^2(G)$, let $g^*(x) = \overline{g(x^{-1})}$, for all $x \in G$. Then $g \longrightarrow g^*$ is a norm preserving conjugate linear bijection of $L^2(G)$ (this is where unimodularity of G is used). Now, for $f, g \in L^2(G)$ and $x \in G$, the following is a standard calculation,

$$\begin{aligned} f*g(x) &= \int_G f(y)g(y^{-1}x)\, dy \\ &= \int_G f(y)\overline{g^*(x^{-1}y)}\, dy \\ &= \int_G f(xy)\overline{g^*(y)}\, dy \\ &= <\lambda_G(x^{-1})f, g^*> . \end{aligned}$$

Thus, $f*g = 0$, for all $f \in F$ if and only if $g^* \in \{\lambda_G(z)f : z\in G, f\in F\}^{\perp} = X(F)^{\perp}$. Therefore, $X(F) = L^2(G)$ if and only if F is not a left zero divisor in $L^2(G)$. $\square$

We are most concerned about the nature of $X(F)$ where $F = \{{\sigma_\alpha}^j\phi : j\in\mathbb{Z}\}$ and ϕ is a refinable function associated with a scaling system.

PROPOSITION 3.4. *Let* (Γ, α) *be a scaling system and let* ϕ *be a refinable function in* $L^2(G)$. *Then* $X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$.

PROOF. According to Proposition 2.2 (c), for any $k\in\mathbb{Z}$, V_k is generated by $\{\lambda_G(\nu)\sigma_\alpha{}^k\phi : \nu\in\alpha^{-k}(\Gamma)\}$. Thus, $V_k\subseteq X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\})$, for all $k\in\mathbb{Z}$. Therefore, $\overline{\bigcup_{j\in\mathbb{Z}} V_j}\subseteq X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\})$. Let $f\in\bigcup_{j\in\mathbb{Z}} V_j$. Then $f\in V_k$, for some k, so $\lambda_G(\nu)f\in V_k$, for all $\nu\in\alpha^{-k}(\Gamma)$. But then $f\in V_j$, for any $j\geq k$, so $\lambda_G(\nu)f\in\bigcup_{j\geq k} V_j$, for any $\nu\in\bigcup_{j\geq k}\alpha^{-j}(\Gamma)$. Because of the nesting properties, $\bigcup_{j\in\mathbb{Z}} V_j$ is invariant under left translations from $\bigcup_{j\in\mathbb{Z}}\alpha^{-j}(\Gamma)$. Therefore, $\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ is also invariant under left translations from $\bigcup_{j\in\mathbb{Z}}\alpha^{-j}(\Gamma)$.

Now, for any $f\in\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ and any $x\in G$, use Proposition 2.1 (d) to select a net (ν_β) of elements from $\bigcup_{j\in\mathbb{Z}}\alpha^{-j}(\Gamma)$ such that $\nu_\beta\to x$ in G. Since λ_G is continuous with respect to the strong operator topology, $\lambda_G(\nu_\beta)f\longrightarrow\lambda_G(x)f$. Therefore, $\lambda_G(x)f\in\overline{\bigcup_{j\in\mathbb{Z}} V_j}$, for all $x\in G$. Hence, $X(\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\}) = \overline{\bigcup_{j\in\mathbb{Z}} V_j}$. □

We are now ready for the main theorem characterizing scaling functions for a scaling system (Γ, α). Recall that, for $\phi\in L^2(G)$ and $j\in\mathbb{Z}$, $\sigma_\alpha{}^j\phi(x) = \delta_\alpha^{j/2}\phi(\alpha^j(x))$ and ϕ is α-substantial if and only if $\{\sigma_\alpha{}^j\phi : j\in\mathbb{Z}\}$ is not a left zero divisor in $L^2(G)$. Combining Proposition 3.3 and 3.4 gives us that $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$ if and only if ϕ is refinable and α-substantial. Thus we have the following theorem.

THEOREM 3.5. *(Density of the union)* Let ϕ be a refinable function in $L^2(G)$ and V_j, $j\in\mathbb{Z}$ defined as above. Then the following are equivalent:

(a) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$

(b) $\{\phi_j\}_{j\in\mathbb{Z}}$ *is a left nonzero divisor in* $L^2(G)$

(c) ϕ *is* α*-substantial.*

Next we show that the condition of being α-substantial is not really that imposing.

PROPOSITION 3.6. *Let* G *be a locally compact group with a dilative automorphism* α. *Let* $f\in L^2(G)$ *satisfy* $f\geq 0$, $f\neq 0$ *and there exists a compact subset* K *of* G *such that* $f(x) = 0$, *for almost all* $x\in G\setminus K$. *Then* f *is* α*-substantial.*

Proof. Since f is compactly supported, it is actually in $L^1(G)$. Without loss of generality, assume $\|f\|_1 = 1$. For each $n\in\mathbb{N}$, define $f_n(x) = \delta_\alpha^n f(\alpha^n(x)) = \delta_\alpha^{n/2}{\sigma_\alpha}^n f(x)$, for all $x\in G$. Then $\int_G f_n(x)\ dx = 1$, for all $n\in\mathbb{N}$. Therefore, for any $g\in L^2(G)$,

$$f_n{*}g(y) - g(y) = \int_G f_n(x)[\lambda_G(x)g(y) - g(y)]dx,$$

for all $y\in G$. Using a version of Minkowski's inequality for integrals instead of sum, see [**5**], VI.11.13 on page 530, we get

$$\begin{aligned}\|f_n{*}g - g\|_2 &= \{\int_G |f_n{*}g(y) - g(y)|^2 dy\}^{1/2} \\ &= \{\int_G |\int_G [\lambda_G(x)g(y) - g(y)]f_n(x)\ dx|^2 dy\}^{1/2} \\ &\leq \int_G \{\int_G |\lambda_G(x)g(y) - g(y)|^2 dy\}^{1/2} f_n(x)dx \\ &= \int_G \|\lambda_G(x)g - g\|_2 f_n(x)dx.\end{aligned}$$

For any $\epsilon > 0$, there exists a neighborhood U of e such that $\|\lambda_G(x)g - g\|_2 < \epsilon$, for all $x\in U$ (this is just the strong operator continuity of λ_G again). Since α is dilative, there exists $n_0\in\mathbb{N}$ such that $\alpha^{-n}(K)\subseteq U$, for all $n\geq n_0$. The support of f_n is contained in $\alpha^{-n}(K)$, so $n\geq n_0$ and $f_n(x)\neq 0$ implies $\|\lambda_G(x)g - g\|_2 < \epsilon$, for almost every $x\in G$. Therefore, $n\geq n_0$ implies

$$\|f_n{*}g - g\|_2 \leq \int_G \|\lambda_G(x)g - g\|_2 f_n(x)\ dx \leq \epsilon \int_G f_n(x)\ dx = \epsilon.$$

Thus, $\{f_n : n = 1, 2, 3, \cdots\}$ forms a left approximate identity for the module action of $L^2(G)$ on $L^2(G)$ by convolution.

Clearly ${\sigma_\alpha}^n f{*}g = 0$ implies $f_n{*}g = 0$. So ${\sigma_\alpha}^n f{*}g = 0$, for all $n\in\mathbb{N}$ implies $g = 0$, for all $g\in L^2(G)$. Thus, $\{{\sigma_\alpha}^j f : j\in\mathbb{Z}\}$ is not a left zero divisor in $L^2(G)$. That is, f is α-substantial. $\square$

4. MRAS GENERATED BY SELF-SIMILAR TILES AND HAAR-LIKE WAVELET BASES

This section concerns refinable functions that arise from self-similar tiles in the space $L^2(G)$, the MRAs generated by a self-similar tile as its scaling function and Haar-like wavelet bases associated with the MRAs. Gröchenig and Madych [**7**] considered the scaling system $(\mathbb{Z}^d, D)$ on the group $\mathbb{R}^d$, where D is a matrix with integer entries with all eigenvalues of which have absolute values bigger than 1. They established a connection between self-similar tilings and MRAs that are generated by a characteristic function for its scaling function. Besides developing the basic properties of self-similar tiles for $(\mathbb{Z}^d, D)$, they looked at a variety of interesting examples by choosing different integer matrices in $\mathbb{R}^2$. For example, the matrix $D = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ has the fractal set known as the twin dragon as a self-similar tile. Self-similar tiles are very often fractal in nature. Following Meyer's recipe, [**7**] also constructed Haar-like wavelet bases using the MRAs generated by self-similar tiles. We have been very much inspired by the results in [**7**].

Let Γ be a uniform lattice in a locally compact group G. A measurable subset T of G is called a tile for G if $m_G(T) < \infty$, $G = \bigcup_{\gamma\in\Gamma} \gamma T$ and $m_G(\gamma T \bigcap T) = 0$, for $\gamma\in\Gamma\backslash\{e\}$. Since Γ is countable, the last condition is equivalent to $\sum_{\gamma\in\Gamma} \chi_T(\gamma^{-1}x) = 1$, for almost all $x\in G$, where χ_A denotes the characteristic function of a subset A of G. The next proposition contains useful observations about tiles.

PROPOSITION 4.1. *Let Γ be a uniform lattice in a locally compact group G. Let T be a tile in G and S a measurable subset of G such that $G = \bigcup_{\gamma\in\Gamma} \gamma S$. Then* (a) *$m_G(T) > 0$,* (b) *$S$ is a tile if and only if $m_G(S) = m_G(T)$.*

PROOF. Since Γ is countable, $m_G(G) = \sum_{\gamma\in\Gamma} m_G(T)$. So $m_G(T) > 0$. For (b), let

$$f(x) = \sum_{\gamma\in\Gamma} \chi_S(\gamma^{-1}x) = \sum_{\gamma\in\Gamma} \chi_{\gamma S}(x) \geq 1,$$

for almost all $x \in G$. Then, for any $\gamma' \in \Gamma$,

$$\begin{aligned} m_G(T) &= \int_{\gamma' T} 1\, dx \leq \int_{\gamma' T} f(x) dx \\ &= \int_G \chi_{\gamma' T}(x) \sum_{\gamma \in \Gamma} \chi_S(\gamma^{-1} x) dx \\ &= \int_G \sum_{\gamma \in \Gamma} \chi_T({\gamma'}^{-1} \gamma x) \chi_S(x) dx \\ &= \int_G \chi_S(x) dx = m_G(S). \end{aligned}$$

Thus, $m_G(T) \leq m_G(S)$ and $m_G(T) = m_G(S)$ if and only if $f(x) = 1$, for almost all $x \in \gamma' T$, for each $\gamma' \in \Gamma$, so, if and only if S is a tile. $\square$

Suppose α is an automorphism of G so that (Γ, α) is a scaling system and T is a tile for G. If $\alpha(T) = \bigcup_{\gamma \in \Gamma_0} \gamma T$, for some subset $\Gamma_0 \subseteq \Gamma$, then we call T a self-similar tile for (Γ, α).

Proposition 4.2. *Let (Γ, α) be a scaling system of a locally compact group G. Suppose that there exists a self-similar tile T for (Γ, α), then the following properties hold:*

(a) *If $\Gamma_0 \subseteq \Gamma$ is such that $\alpha(T) = \bigcup_{\gamma \in \Gamma_0} \gamma T$, then Γ_0 is a complete set of right coset representatives for $\alpha(\Gamma)$ in Γ,*

(b) *$[\Gamma : \alpha(\Gamma)] = \delta_\alpha$,*

(c) *$\phi = m_G(T)^{-1/2} \chi_T$ is a refinable function in $L^2(G)$.*

Proof. We begin by proving (b). Let $\gamma_1, \cdots, \gamma_k$ be a complete set of right coset representations for $\alpha(\Gamma)$ in Γ. So Γ is the disjoint union $\bigcup_{i=1}^k \alpha(\Gamma)\gamma_i$.

Since T is a tile for G,

$$G = \bigcup_{\gamma \in \Gamma} \alpha(\gamma)[\bigcup_{i=1}^k \gamma_i T].$$

Applying α^{-1}, we get

$$G = \bigcup_{\gamma \in \Gamma} \gamma \alpha^{-1}[\bigcup_{i=1}^k \gamma_i T] = \bigcup_{\gamma \in \Gamma} \gamma [\bigcup_{i=1}^k \alpha^{-1}(\gamma_i T)].$$

If $S = \bigcup_{i=1}^{k} \alpha^{-1}(\gamma_i T)$, then one easily checks that $m_G(\gamma S \bigcap S) = 0$ if $\gamma \neq e$. Thus S is a tile. So $m_G(S) = m_G(T)$. On the other hand, $m_G(S) = k\delta_\alpha^{-1} m_G(T)$. Therefore, $[\Gamma : \alpha(\Gamma)] = k = \delta_\alpha$ and (b) holds.

For (a), suppose Γ_0 is the subset of Γ such that $\alpha(T) = \bigcup_{\gamma \in \Gamma_0} \gamma T$. Then $\chi_{\alpha(T)}(x) = \sum_{\gamma' \in \Gamma_0} \chi_{\gamma' T}(x)$, for almost all x. Since G is "tiled" by the sets $\{\gamma T : \gamma \in \Gamma\}$, the $\{\chi_{\gamma T} : \gamma \in \Gamma\}$ are mutually orthogonal projections in the commutative von Newmann algebra $L^\infty(G)$ (at least, that is a fancy way of thinking for the following calculation). Each of the following equalities is true for almost every $x \in G$.

$$\begin{aligned}
1 &= \sum_{\gamma \in \Gamma} \chi_{\gamma T}(x) = \sum_{\gamma \in \Gamma} \chi_{\gamma T}(\alpha^{-1}(x)) \\
&= \sum_{\gamma \in \Gamma} \chi_{\alpha(\gamma)\alpha(T)}(x) = \sum_{\nu \in \alpha(\Gamma)} \chi_{\nu \alpha(T)}(x) \\
&= \sum_{\nu \in \alpha(\Gamma)} \chi_{\alpha(T)}(\nu^{-1} x) = \sum_{\nu \in \alpha(\Gamma)} \sum_{\gamma' \in \Gamma_0} \chi_{\gamma' T}(\nu^{-1} x) \\
&= \sum_{\nu \in \alpha(\Gamma)} \sum_{\gamma' \in \Gamma_0} \chi_{(\nu \gamma')T}(x).
\end{aligned}$$

Thus, $\sum_{\gamma \in \Gamma} \chi_{\gamma T}(x) = \sum_{\nu \in \alpha(\Gamma)} (\sum_{\gamma' \in \Gamma_0} \chi_{(\nu\gamma')T}(x))$, for almost every $x \in G$. This implies that each $\gamma \in \Gamma$ has a unique expression of the form $\nu \gamma'$, with $\nu \in \alpha(\Gamma)$ and $\gamma' \in \Gamma_0$. In other words, Γ_0 is a complete set of right coset representations for $\alpha(\Gamma)$ in Γ.

Finally, we prove part (c). If $\phi = m_G(T)^{-1/2} \chi_T$, then $\|\phi\|_2 = 1$, then ϕ has orthogonal shifts and we can see that ϕ is refinable as follows. For any $x \in G$,

$$\begin{aligned}
\sigma_\alpha^{-1} \phi(x) &= \delta_\alpha^{-1/2} \phi(\alpha^{-1}(x)) \\
&= \delta_\alpha^{-1/2} m_G(T)^{-1/2} \chi_T(\alpha^{-1}(x)) \\
&= \delta_\alpha^{-1/2} m_G(T)^{-1/2} \chi_{\alpha T}(x) \\
&= \sum_{\gamma \in \Gamma_0} \delta_\alpha^{-1/2} m_G(T)^{-1/2} \chi_{\gamma T}(x) \\
&= \sum_{\gamma \in \Gamma_0} \delta_\alpha^{-1/2} m_G(T)^{-1/2} \chi_T(\gamma^{-1} x) \\
&= \sum_{\gamma \in \Gamma_0} \delta_\alpha^{-1/2} \lambda_G(\gamma) \phi(x).
\end{aligned}$$

So ${\sigma_\alpha}^{-1}\phi = \sum_{\gamma\in\Gamma_0} {\delta_\alpha^{-1/2}}\lambda_G(\gamma)\phi$ which implies $\phi = \sum_{\gamma\in\Gamma_0} {\delta_\alpha^{-1/2}}\sigma_\alpha[\lambda_G(\gamma)\phi]$. Therefore ϕ is refinable. □

From the proof above, we see that the number of right coset representatives for $\alpha(\Gamma)$ in Γ is equal to δ_α. This number will appear later on.

Boor, DeVore and Ron in [**2**] showed that refinability is not enough to generate an MRA in the space $L^2(\mathbb{R}^d)$. Using the results from section 3, we see that whenever we have a refinable function of self-similar tile, an MRA can always be produced by this function as a scaling function in the space $L^2(G)$.

THEOREM 4.3. Let G be a locally compact group and (Γ, α) a scaling system on G. Suppose that there exists a self-similar tile T for (Γ, α) on G. Then $\phi = \chi_T$ is a scaling function, that is, it will generate an MRA for the space $L^2(G)$.

PROOF. The refinability of ϕ guarantees that condition (iii) in the definition holds. Define $V_0 = V(\phi)$, the closure of $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$. It is clear that $\{\lambda_G(\gamma)\phi : \gamma \in \Gamma\}$ is an orthonormal basis for V_0. Thus condition (i) holds. Using the unitary operator σ_α, a sequence of closed subspaces $V_j = \sigma_\alpha^j V_0$ are constructed. Condition (iv) is trivial by Theorem 3.1. By Proposition 3.6, ϕ is α-substantial. Thus the density of the union (v) is also satisfied. Therefore, an MRA for $L^2(G)$ is generated by the self-similar tile χ_T as a scaling function. □

Once an MRA has been built up in the space $L^2(G)$, next we want to construct wavelet basis using the structure provided by the MRA.

Let $\{V_j : j\in\mathbb{Z}\}$ be an MRA in the space $L^2(G)$ with a self-similar tile χ_T as its scaling function for the scaling system (Γ, α). Let W_j be the orthogonal complement V_j in V_{j+1}, that is, $V_{j+1} = V_j \bigoplus W_j$, $j\in\mathbb{Z}$. Then we can decompose $L^2(G)$ as $\bigoplus_{j\in\mathbb{Z}} W_j$. To construct an orthogonal wavelet basis for $L^2(G)$, all we need is to construct an orthogonal basis for W_0. If an orthogonal basis for W_0 can be constructed, then σ_α^j will send this orthogonal basis for W_0 to an orthogonal basis for $W_j, j\in\mathbb{Z}$. Therefore the union of all these bases would give an orthogonal basis for $L^2(G)$ because $L^2(G) = \bigoplus_{j\in\mathbb{Z}} W_j$. In the space $L^2(\mathbb{R}^d)$ with a scaling system

$(\mathbb{Z}^d, D)$, if ϕ is a scaling function of an MRA and $q = |det(D)|$, [**9**] showed that there exist $q-1$ functions $\psi_1, \cdots, \psi_{q-1}$ such that $\{T_k\psi_i : k\in\mathbb{Z}^d, i = 1, \cdots, q-1\}$ is an orthogonal basis of W_0, where $\psi_i, i = 1, \cdots, q-1$ satisfy

$$\psi_i(x) = \sum_{k\in\mathbb{Z}^d} a_{ik}|det(D)|^{1/2}\phi(Dx-k), \tag{4.1}$$

with some sequences $\{a_{ik}\}, i = 1, \cdots, q-1$ in $l^2(\mathbb{Z}^d)$. Therefore, $\{\sigma_D^j T_k\psi_i : k\in\mathbb{Z}^d, j\in\mathbb{Z}, i = 1, \cdots, q-1\}$ forms an orthogonal wavelet basis for $L^2(\mathbb{R}^d)$. Since $V_0\subset V_1$, there must exist a sequence $\{a_k\}$ in $l^2(\mathbb{Z}^d)$ such that

$$\phi(x) = \sum_{k\in\mathbb{Z}^d} a_k\sigma_D T_k\phi(x) = \sum_{k\in\mathbb{Z}^d} a_k|det(D)|^{1/2}\phi(Dx-k). \tag{4.2}$$

To construct a wavelet basis following Meyer's recipe, one first begins with an MRA with a scaling function ϕ satisfying equation (4.2) and then look for wavelet basis satisfying equation (4.1). For the MRA in $L^2(\mathbb{R}^d)$ generated by a self-similar tile as a scaling function, [**7**] constructed a piecewise constant wavelet basis associated with the scaling system $(\mathbb{Z}^d, D)$ following Meyer's recipe (See [**9**]). It turns out that Meyer's recipe still works in the space $L^2(G)$ if an MRA generated by a scaling function of self-similar tile is available.

Proposition 4.4. *Let χ_T be a self-similar tile associated with a scaling system (Γ, α) and $\{V_j : j\in\mathbb{Z}\}$ be an MRA generated by χ_T. Let $\Gamma_0 = \{\gamma_1, \gamma_2, \cdots, \gamma_{\delta_\alpha}\}$ be a complete set of right coset representatives for $\alpha(\Gamma)$ in Γ. Then the subspace W_0 is a set of functions satisfying $f(x) = \sum_{\gamma\in\Gamma} a_\gamma\sigma_\alpha\lambda_G(\gamma)\chi_T(x)$ with $\{a_\gamma\}$ in $l^2(\Gamma)$ satisfying $\sum_{\gamma'\in\Gamma_0} a_{\alpha(\gamma)\gamma'} = 0$ for all $\gamma\in\Gamma$.*

Proof. A function $f\in W_0\subset V_1$ can be written as

$$f(x) = \sum_{\gamma\in\Gamma} a_\gamma\sigma_\alpha\lambda_G(\gamma)\chi_T(x) = \sum_{\gamma\in\Gamma} a_\gamma\delta_\alpha^{1/2}\chi_T(\gamma^{-1}\alpha(x)),$$

for some sequence $\{a_\gamma\}$ in $l^2(\Gamma)$. A function $g\in V_0$ can be written as

$$g(x) = \sum_{\gamma'\in\Gamma} b_{\gamma'}\lambda_G(\gamma')\chi_T(x) = \sum_{\gamma'\in\Gamma} b_{\gamma'}\chi_T({\gamma'}^{-1}x),$$

for some sequence $\{b_{\gamma'}\}$ in $l^2(\Gamma)$. Then

$$\begin{aligned}
\langle f,g\rangle &= \int_G \sum_{\gamma\in\Gamma} a_\gamma \delta_\alpha^{1/2}\chi_T(\gamma^{-1}\alpha(x))\overline{\sum_{\gamma'\in\Gamma} b_{\gamma'}\chi_T({\gamma'}^{-1}x)}dx \\
&= \delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}\int_G \chi_{\alpha^{-1}(\gamma T)}(x)\chi_{\gamma' T}(x)dx \\
&= \delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}m_G(\alpha^{-1}(\gamma T)\bigcap\gamma' T) \\
&= \delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}m_G(\alpha^{-1}(\gamma)\alpha^{-1}(T)\bigcap\gamma' T).
\end{aligned}$$

Since $\Gamma=\bigcup_{i=1}^{\delta_\alpha}\alpha(\Gamma)\gamma_i$, which is a union of disjoint right cosets, a sum over the set Γ is equal to the sum over the set $\bigcup_{i=1}^{\delta_\alpha}\alpha(\Gamma)\gamma_i$. Thus the above equals

$$\begin{aligned}
&\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{\gamma'\in\Gamma} a_\gamma\overline{b_{\gamma'}}m_G(\alpha^{-1}(\gamma)\alpha^{-1}(T)\bigcap\gamma' T) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i)\alpha^{-1}(T)\bigcap\gamma' T) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma')\alpha(T))) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma')\bigcup_{j=1}^{\delta_\alpha}\gamma_j T)) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha}\sum_{\gamma'\in\Gamma} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma'}}m_G(\alpha^{-1}(\bigcup_{j=1}^{\delta_\alpha}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma')\gamma_j T))) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma}}m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T)) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma}\sum_{i=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_i}\overline{b_{\gamma}}m_G(\alpha^{-1}(T)) \\
=\ &\delta_\alpha^{1/2}\sum_{\gamma\in\Gamma} c_\gamma\overline{b_{\gamma}}m_G(\alpha^{-1}(T)),
\end{aligned}$$

where $c_\gamma=\sum_{i=1}^{\delta_\alpha}a_{\alpha(\gamma)\gamma_i}$. The third last equality is due to the following basic fact: $m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T\bigcap\alpha(\gamma)\gamma_j T))$ is either equal to 0 or $m_G(\alpha^{-1}(\alpha(\gamma)\gamma_i T))$ because T is a tile. The second last equality holds because G is unimodular. Thus, $\langle f,g\rangle = \delta_\alpha^{1/2}\langle c,b\rangle\delta_\alpha^{-1}=\delta_\alpha^{-1/2}\langle c,b\rangle$. Therefore, $f\in W_0$ if and only if f can be written as $f(x)=\sum_{\gamma\in\Gamma}a_\gamma\delta_\alpha^{1/2}\chi_T(\gamma^{-1}\alpha(x))$ and $c_\gamma=\sum_{i=1}^{\delta_\alpha}a_{\alpha(\gamma)\gamma_i}=0$. □

Proposition 4.5. *Suppose that χ_T, (Γ,α), $\{V_j : j\in\mathbb{Z}\}$, and Γ_0 are the same as in Proposition 4.4. Suppose that $U = (u_{ij})$ is a $\delta_\alpha \times \delta_\alpha$ unitary matrix with all entries on the first row being the same constant $\delta_\alpha^{-1/2}$. Then the set of functions $\{\psi_1, \psi_2, \cdots, \psi_{\delta-1}\}$ defined by*

$\psi_{i-1}(x) = m_G(T)^{-1/2}\sum_{j=1}^{\delta_\alpha} u_{ij}\sigma_\alpha\lambda_G(\gamma_j)\chi_T(x), i = 2, \cdots, \delta_\alpha$

is a set of mother wavelets for the MRA. That is, the following set

$F = \{\lambda_G(\gamma)\psi_i : \gamma\in\Gamma, i = 1, \cdots, \delta_\alpha - 1\}$

is a complete orthonormal basis for the space W_0.

Proof. We first show that the set F is an orthogonal system and then prove it is complete.

In the following, we will use Proposition 2.2 (a): $\lambda_G(\gamma)\sigma_\alpha = \sigma_\alpha\lambda_G(\alpha(\gamma))$. For $\lambda_G(\gamma')\psi_{i-1}, \lambda_G(\gamma'')\psi_{j-1}\in F$, then

$$\begin{aligned}
&\langle\lambda_G(\gamma')\psi_{i-1}, \lambda_G(\gamma'')\psi_{j-1}\rangle\\
= \;& \langle\lambda_G(\gamma')m_G(T)^{-1/2}\sum_{m=1}^{\delta_\alpha} u_{im}\sigma_\alpha\lambda_G(\gamma_m)\chi_T, \lambda_G(\gamma'')m_G(T)^{-1/2}\sum_{n=1}^{\delta_\alpha} u_{jn}\sigma_\alpha\lambda_G(\gamma_n)\chi_T\rangle\\
= \;& m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\gamma')\sigma_\alpha\lambda_G(\gamma_m)\chi_T, \lambda_G(\gamma'')\sigma_\alpha\lambda_G(\gamma_n)\chi_T\rangle\\
= \;& m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\gamma')\sigma_\alpha\lambda_G(\gamma_m)\chi_T, \lambda_G(\gamma'')\sigma_\alpha\lambda_G(\gamma_n)\chi_T\rangle\\
= \;& m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\sigma_\alpha\lambda_G(\alpha(\gamma'))\lambda_G(\gamma_m)\chi_T, \sigma_\alpha\lambda_G(\alpha(\gamma''))\lambda_G(\gamma_n)\chi_T\rangle\\
= \;& m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\alpha(\gamma'))\lambda_G(\gamma_m)\chi_T, \lambda_G(\alpha(\gamma''))\lambda_G(\gamma_n)\chi_T\rangle\\
= \;& m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}\langle\lambda_G(\alpha(\gamma')\gamma_m)\chi_T, \lambda_G(\alpha(\gamma'')\gamma_n)\chi_T\rangle\\
= \;& m_G(T)^{-1}\sum_{m=1}^{\delta_\alpha}\sum_{n=1}^{\delta_\alpha} u_{im}\overline{u_{jn}}m_G(\alpha(\gamma')\gamma_m T\cap\alpha(\gamma'')\gamma_n T)\\
= \;& m_G(T)^{-1}\delta(\gamma'-\gamma'')\sum_{m=1}^{\delta_\alpha} u_{im}\overline{u_{jm}}m_G(T)\\
= \;& \delta(\gamma'-\gamma'')\delta(i-j).
\end{aligned}$$

Next we show that the set F is complete in W_0. For any $f\in W_0$, we want to prove that if $\langle f, \lambda_G(\gamma)\psi_{i-1}\rangle = 0$ for any $\gamma\in\Gamma$, $i = 2, 3, \cdots, \delta_\alpha$, then $f = 0$. We see that

$$\begin{aligned}
\langle f, \lambda_G(\gamma)\psi_i\rangle &= \langle \sum_{\gamma'\in\Gamma} a_{\gamma'}\sigma_\alpha\lambda_G(\gamma')\chi_T, \lambda_G(\gamma)m_G(T)^{-1/2}\sum_{j=1}^{\delta_\alpha} u_{ij}\sigma_\alpha\lambda_G(\gamma_j)\chi_T\rangle \\
&= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}\langle\sigma_\alpha\lambda_G(\gamma')\chi_T, \lambda_G(\gamma)\sigma_\alpha\lambda_G(\gamma_j)\chi_T\rangle \\
&= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}\langle\sigma_\alpha\lambda_G(\gamma')\chi_T, \sigma_\alpha\lambda_G(\alpha(\gamma))\lambda_G(\gamma_j)\chi_T\rangle
\end{aligned}$$

$$\begin{aligned}
&= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}\langle\lambda_G(\gamma')\chi_T, \lambda_G(\alpha(\gamma)\gamma_j)\chi_T\rangle \\
&= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{j=1}^{\delta_\alpha} a_{\gamma'}\overline{u_{ij}}m_G(\gamma'T\bigcap\alpha(\gamma)\gamma_jT) \\
&= m_G(T)^{-1/2}\sum_{\gamma'\in\Gamma}\sum_{m=1}^{\delta_\alpha}\sum_{j=1}^{\delta_\alpha} a_{\alpha(\gamma')\gamma_m}\overline{u_{ij}}m_G(\alpha(\gamma')\gamma_mT\bigcap\alpha(\gamma)\gamma_jT) \\
&= m_G(T)^{-1/2}\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}}m_G(\alpha(\gamma)\gamma_mT) \\
&= m_G(T)^{1/2}\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}}.
\end{aligned}$$

Thus, $\langle f, \lambda_G(\gamma)\psi_{i-1}\rangle = 0$ for any $\gamma\in\Gamma$, $i = 2, 3, \cdots, \delta_\alpha$ means $\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}} = 0$ for any $\gamma\in\Gamma$, $i = 2, 3, \cdots, \delta_\alpha$. Since $f\in W_0$ and all entries on the first row of the unitary matrix $U = (u_{ij})$ are constant, proposition 4.4 implies that $\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{1m}} = 0$ for any $\gamma\in\Gamma$. Thus, $\sum_{m=1}^{\delta_\alpha} a_{\alpha(\gamma)\gamma_m}\overline{u_{im}} = 0$ for any $\gamma\in\Gamma$, $i = 1, 2, 3, \cdots, \delta_\alpha$. This shows that, for any $\gamma\in\Gamma$, the vector $(a_{\alpha(\gamma)\gamma_1}, a_{\alpha(\gamma)\gamma_2}, \cdots, a_{\alpha(\gamma)\gamma_{\delta_\alpha}})$ is perpendicular to all rows in the unitary matrix U. So, $a_{\alpha(\gamma)\gamma_m}$ must be 0 for any $\gamma\in\Gamma$ and $m = 1, 2, \cdots, \delta_\alpha$. Therefore, $a_\gamma = 0$ for any $\gamma\in\Gamma$. Hence $f = 0$. That is, The set F is complete in W_0. □

Theorem 4.6. Given $\psi_{i-1}, i = 2, \cdots, \delta_\alpha$ that are defined in Proposition 4.5, the set $\{\sigma_\alpha^j\psi_i : j\in\mathbb{Z}, i = 1, 2, \cdots, \delta_\alpha - 1\}$ forms a complete orthonormal basis for $L^2(G)$.

5. EXAMPLES ON THE HEISENBERG GROUP

The theorems in sections 3 and 4 hold for general space $L^2(G)$, where G is a locally compact group which includes the Heisenberg group as an important example. In this section, we show examples to illustrate those theorems. All we need to do is to construct the refinable functions of self-similar tile on the Heisenberg group. According to the theorems in sections 3 and 4, the existence of refinable functions of self-similar tile will automatically lead us to build MRAs, hence to create Haar-like wavelet bases on the Heisenberg group.

Let G be the $2d+1$ dimensional Heisenberg group $\mathbb{H}^d$, which is a nilpotent Lie group with underlying manifold $\mathbb{R}^{2d+1}$. We denote points in $\mathbb{H}^d$ by $(\underline{q}, \underline{p}, t)$ with $\underline{q}, \underline{p} \in \mathbb{R}^d$, $t \in \mathbb{R}$, and define the group operation by $(\underline{q}, \underline{p}, t)(\underline{q}', \underline{p}', t') = (\underline{q} + \underline{q}', \underline{p} + \underline{p}', t + t' + \frac{1}{2}(\underline{p} \cdot \underline{q}' - \underline{p}' \cdot \underline{q}))$. Let Γ be the following uniform lattice subgroup in $\mathbb{H}^d$: $\Gamma = \{ (\underline{m}, \underline{n}, l/2) : \underline{m}, \underline{n} \in \mathbb{Z}^d, l \in \mathbb{Z}\}$. And let α be a dilative automorphism given by $\alpha(\underline{q}, \underline{p}, t) := (2\underline{q}, 2\underline{p}, 2^2 t)$. Then (Γ, α) forms a scaling system on $\mathbb{H}^d$ with $\delta = [\Gamma : \alpha(\Gamma)] = 2^{2(d+1)}$.

It is known (Folland [**6**]) that every automorphism α of $\mathbb{H}^d$ can be uniquely decomposed as a product of four factors $\alpha_1\alpha_2\alpha_3\alpha_4$, with $\alpha_j \in G_j$ $(j = 1, 2, 3, 4)$, where G_j is defined as follows: G_1 denotes the symplectic group $\mathrm{ps}(d, \mathbb{R})$; G_2 consists of inner automorphisms: $(\underline{a}, \underline{b}, c)(\underline{q}, \underline{p}, t)(\underline{a}, \underline{b}, c)^{-1} = (\underline{q}, \underline{p}, c + \underline{a} \cdot \underline{p} - \underline{b} \cdot \underline{q})$; G_3 consists of dilations $\delta[r]$ defined by $\delta[r](\underline{q}, \underline{p}, t) = (r\underline{q}, r\underline{p}, r^2 t)$; and G_4 consists of two elements, the identity and the automorphism i defined by $i(\underline{q}, \underline{p}, t) = (\underline{p}, \underline{q}, -t)$.

We restrict ourself to constructing special self-similar tiles for (Γ, α) on $\mathbb{H}^d$, where α can be written as $\alpha_1\alpha_3\alpha_4$. That is, $\alpha(\underline{q}, \underline{p}, t) = (D_\alpha(\underline{q}, \underline{p}), r_\alpha t)$, where r_α is some integer and D_α is a dilative automorphism from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$. The fundamental idea to construct such self-similar tiles for (Γ, α) is the following. We decompose the process of construction into two steps: first constructing in the direction $\mathbb{R}^{2d}$, that is, constructing self-similar tiles for the scaling system $(\mathbb{Z}^{2d}, D_\alpha)$. The work in [**7**] provides us details for this. Then, based on the self-similar tile obtained for $(\mathbb{Z}^{2d}, D_\alpha)$, we construct a self-similar tile in the direction $\mathbb{R}$ for (Γ, α). Such a self-similar tile is called a self-similar stacked tile due to the obvious geometric reason.

For simple notation reason, let's use $(\underline{x}, t)$ to denote the element $(\underline{q}, \underline{p}, t)$ in the Heisenberg group, that is, $\underline{x} = (\underline{q}, \underline{p}) \in \mathbb{R}^{2d}$. Then the group law becomes $(\underline{x}, t)(\underline{x'}, t') = (\underline{x} + \underline{x'}, t + t' + S(\underline{x}, \underline{x'}))$ where $S(\underline{x}, \underline{x'}) = ((\underline{q}, \underline{p}), (\underline{q'}, \underline{p'})) = 1/2(\underline{p} \cdot \underline{q'} - \underline{q} \cdot \underline{p'})$ is a skew-symmetric bilinear form from $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$ to $\mathbb{R}$.

Let A be a self-similar tile for the scaling system $(\mathbb{Z}^{2d}, D_\alpha)$ on $\mathbb{R}^{2d}$. The existence of A is confirmed by [**7**]. Such an A is measurable. Without loss of generality, we can assume that

$$A \bigcap (\underline{k} + A) = \emptyset \text{ for } \underline{k} \neq 0,\ \underline{k} \in \mathbb{Z}^{2d} \text{ and } \bigcup_{\underline{k} \in \mathbb{Z}^{2d}} (\underline{k} + A) = \mathbb{R}^{2d}, \text{ and } D_\alpha(A) = \bigcup_{i=1}^{s} (\underline{k}_i + A)$$

where $k_1,\ k_2,\ \cdots,\ k_s$ are lattice points that are representatives of distinct cosets in $\mathbb{Z}^{2d}/D_\alpha(\mathbb{Z}^{2d})$. Thus, the Lebesgue measure of A must be 1, see lemma 1 in [**7**]. Since the measure of A is 1 and the disjoint union $\bigcup_{k \in \mathbb{Z}^{2d}} (k + A)$ fill out the whole space $\mathbb{R}^{2d}$, we could arrange a one to one correspondence between the lattice points in $\mathbb{Z}^{2d}$ and the tiles. Or simply speaking, we can assume that each tile only contains one lattice point. For $\underline{x} \in \mathbb{R}^{2d}$, we use $[\underline{x}]_A$ to denote the lattice point that corresponds to the tile which contains $\underline{x}$. Let $< \underline{x} >_A = \underline{x} - [\underline{x}]_A \in A$.

Let F be a bounded measurable real-valued function defined first on A and then extended periodically to the whole space $\mathbb{R}^{2d}$. Thus, we have $F(\underline{x}) = F(< \underline{x} >_A)$. We are going to produce a self-similar tile, denoted by T, for the scaling system (Γ, α) as follows: $T = \{\ (\underline{x}, t) \in \mathbb{H}^d\ :\ \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2\ \}$, where F is to be determined later. We can view $F(\underline{x})$ as a piece of surface over A and think of T as a solid over A bounded between two surfaces $F(\underline{x})$ and $F(\underline{x}) + 1/2$. Thus the volume of T is equal to $1/2$. So we can think of the "thickness" (in the direction of t-axis) of tile T as $1/2$.

For an element $\gamma = (\underline{a}, l/2) \in \Gamma$, the image of T under the left translation by γ is given by $\gamma T = \{\ (\underline{x}, t) \in \mathbb{H}^d\ :\ \underline{x} - \underline{a} \in A, 0 \leq t - l/2 - S(\underline{a}, \underline{x} - \underline{a}) - F(\underline{x}) < 1/2\ \}$. To show that $\bigcup_{\gamma \in \Gamma} \gamma T$ is a tiling of $\mathbb{H}^d$, we need to check two things. (a) $\bigcup_{\gamma \in \Gamma} \gamma T$ is a disjoint union. (b) $\bigcup_{\gamma \in \Gamma} \gamma T$ fills out the whole space $\mathbb{H}^d$. For (a), if $\underline{a} \neq \underline{a'}$, then

$(\underline{a}, l/2)T\bigcap(\underline{a'}, l/2)T = \emptyset$ since the image $(\underline{a}, l/2)T$ of T is in a stack of tiles lying over the tile $(\underline{a}, 0)T$. If l and l' are different integers, then $(\underline{a}, l/2)T$ and $(\underline{a}, l'/2)T$ are two different tiles in one stack located at tile $(\underline{a}, 0)T$, but $(\underline{a}, l/2)T\bigcap(\underline{a}, l'/2)T = \emptyset$ since the thickness for each tile is $1/2$. As for (b), for any $(\underline{x}, t)\in\mathbb{H}^d$, there exists a unique element $\underline{a}\in\mathbb{Z}^{2d}$ such that $\underline{x} - \underline{a}\in A$. And also there exists a unique element $l\in\mathbb{Z}$ with the property $0\leq t - l/2 - S(\underline{a}, \underline{x} - \underline{a}) - F(\underline{x} - \underline{a}) < 1/2$.

Now, we can start constructing a self-similar stacked tiling related to the tile A in $\mathbb{R}^{2d}$. From the explanation above, we know that the key point is to determine the surface described by the equation $t = F(\underline{x})$ on A. We start by choosing

$$\Gamma_0 = \{ \ (\underline{k_i}, c) \ : \ i = 1, 2, \cdots, s, \text{ and } c = 0, 1/2, 1, 3/2, \cdots, (|r_\alpha| - 1)/2 \ \}.$$

Then we have

Proposition 5.1. *T is a self-similar stacked tile for (Γ, α) with the above choice of the finite set Γ_0 if and only if the function $F(\underline{x})$ on A satisfies*

$$F(\underline{x}) = \frac{1}{|r_\alpha|}F(< D_\alpha(\underline{x}) >_A) + \frac{1}{|r_\alpha|}S([D_\alpha(\underline{x})]_A, < D_\alpha(\underline{x}) >_A).$$

Proof. By the choice of Γ_0, we have

$$\begin{aligned} & \bigcup\nolimits_{\gamma\in\Gamma_0}\gamma T \text{ (disjoint finite union)} \\ = \ & \{ \ (\underline{x}, t) : \underline{x}\in\bigcup\nolimits_{i=1}^{s}(\underline{k_i} + A), 0\leq t - S([\underline{x}]_A, < \underline{x} >_A) - F(< \underline{x} >_A) < \frac{|r_\alpha|}{2} \ \} \\ = \ & \{(\underline{x}, t) : \underline{x}\in\bigcup\nolimits_{i=1}^{s}(\underline{k_i} + A), 0\leq\frac{1}{|r_\alpha|}t - \frac{1}{|r_\alpha|}S([\underline{x}]_A, < \underline{x} >_A) - \frac{1}{|r_\alpha|}F(< \underline{x} >_A) < \frac{1}{2}\}. \end{aligned}$$

Geometrically speaking, there are s stacks of tiles in $\bigcup_{\gamma\in\Gamma_0}\gamma T$. For each stack there are $|r_\alpha|$ tiles with the "thickness" for each tile $1/2$, so the "thickness" for each stack is $|r_\alpha|\times 1/2$. On the other hand,

$$\begin{aligned} \alpha T &= \alpha\{ \ (\underline{x}, t)\in\mathbb{H}^d \ : \ \underline{x}\in A, 0\leq t - F(\underline{x}) < 1/2 \ \} \\ &= \{ \ (D_\alpha(\underline{x}), r_\alpha t)\in\mathbb{H}^d \ : \ \underline{x}\in A, 0\leq t - F(\underline{x}) < 1/2 \ \} \\ &= \{ \ (\underline{x}, t)\in\mathbb{H}^d \ : \ {D_\alpha}^{-1}(\underline{x})\in A, 0\leq\frac{t}{|r_\alpha|} - F(D_\alpha^{-1}(\underline{x})) < 1/2 \ \} \\ &= \{ \ (\underline{x}, t)\in\mathbb{H}^d \ : \ \underline{x}\in D_\alpha(A) = \bigcup\nolimits_{i=1}^{s}(\underline{k_i} + A) \text{ and } 0\leq\frac{t}{|r_\alpha|} - F(D_\alpha^{-1}(\underline{x})) < 1/2 \ \}. \end{aligned}$$

These two sets are equal if and only if

$$F(D_\alpha^{-1}(\underline{x})) = \frac{1}{|r_\alpha|}F(<\underline{x}>_A) + \frac{1}{|r_\alpha|}S([\underline{x}]_A, <\underline{x}>_A).$$

Or equivalently

$$F(\underline{x}) = \frac{1}{|r_\alpha|}F(<D_\alpha(\underline{x})>_A) + \frac{1}{|r_\alpha|}S([D_\alpha(\underline{x})]_A, <D_\alpha(\underline{x})>_A).$$

□

This proposition yields the following theorem.

THEOREM 5.2. For the choice of Γ_0 given above, there exists a unique self-similar stacked tile T for (Γ, α). The function $F(\underline{x})$ is given explicitly by

$$F(\underline{x}) = \sum_{m=1}^{\infty} \frac{1}{|r_\alpha|^m} S([D_\alpha^m(\underline{x})]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), <D_\alpha^m(\underline{x})>_A),$$

where a lattice point $\underline{k}$ mod $(D_\alpha(\mathbb{Z}^{2d}))$ equals the representative of the coset which contains element $\underline{k}$.

PROOF. Define a mapping M from $L^\infty(A)$ to $L^\infty(A)$ by

$$Mf(\underline{x}) = \frac{1}{|r_\alpha|}F(<D_\alpha(\underline{x})>_A) + \frac{1}{|r_\alpha|}S([D_\alpha(\underline{x})]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), <D_\alpha(\underline{x})>_A),$$

where $L^\infty(A)$ is a Banach space with the supermum norm. Given $f,\ g \in L^\infty(A)$, we have

$$\begin{aligned} \|Mf - Mg\|_{L^\infty(A)} &= \|\frac{1}{|r_\alpha|}f(<D_\alpha(\underline{x})>_A) - \frac{1}{|r_\alpha|}g(<D_\alpha(\underline{x})>_A)\|_{L^\infty(A)} \\ &\leq \frac{1}{|r_\alpha|}\|f-g\|_{L^\infty(A)}. \end{aligned}$$

So M is a contractive mapping. There exists a unique fixed point, denoted by $F(\underline{x})$. Especially, we have $F = \lim_{m\to\infty} M^m 0$. Thus,

$$F(\underline{x}) = \sum_{m=1}^{\infty} \frac{1}{|r_\alpha^m|} S([D_\alpha^m(\underline{x})]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), <D_\alpha^m(\underline{x})>_A).$$

□

Now we can provide the first example based on Theorem 5.2.

EXAMPLE 5.3. Consider α from $\mathbb{H}^d$ to $\mathbb{H}^d$ defined by $\alpha(\underline{q}, \underline{p}, t) := (2\underline{q}, 2\underline{p}, 2^2 t)$. It is clear that α is in G_3. We can write $\alpha(\underline{q}, \underline{p}, t) = (D_\alpha(\underline{q}, \underline{p}), r_\alpha t) = (2(\underline{q}, \underline{p}), 4t)$. Thus, $r_\alpha = 4$ and D_α is the dilative automorphism on $\mathbb{R}^{2d}$. Let $A = \{ \ \underline{x} \in \mathbb{R}^{2d} \ : \ 0 \leq x_j < 1, j = 1, 2, \cdots, 2n \ \}$ denote the "half open and half closed" standard tile in the Euclidean space $\mathbb{R}^{2d}$, where x_j denotes the jth component of $\underline{x}$. Then it is obvious that $\bigcup_{\underline{a} \in \mathbb{Z}^{2d}} (A + \underline{a})$ (disjoint union) fills out the whole space $\mathbb{R}^{2d}$. Clearly, A is a self-similar tile. If we choose $\Gamma_0 = \{ \ (\underline{a}, b) \ : \ a_j = 0 \text{ or } 1, 1 \leq j \leq 2n, b = 0, 1/2, 1 \text{ or } 3/2 \ \}$, then by Theorem 5.2, $T = \{ \ (\underline{x}, t) \in \mathbb{H}^d \ : \ \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2 \ \}$ is a self similar-tile for (Γ, α) with F defined by

$$\begin{aligned} F(\underline{x}) &= \sum_{m=1}^{\infty} \frac{1}{4^m} S([D_\alpha^m(\underline{x})]_A \text{mod } (D_\alpha^m(\mathbb{Z}^{2d})), < D_\alpha^m(\underline{x}) >_A) \\ &= \sum_{m=1}^{\infty} \frac{1}{4^m} S([2^m \underline{x}] \text{mod} 2, < 2^m \underline{x} >), \end{aligned}$$

where $[2^m \underline{x}]$mod 2 means $([2^m x_1]\text{mod } 2, [2^m x_2]\text{mod } 2, \cdots, [2^m x_{2n}]\text{mod } 2)$.

EXAMPLE 5.4. In this example, we choose a different dilative automorphism on $\mathbb{H}^d$ which is defined as follows. $\alpha(\underline{q}, \underline{p}, t) := (2\underline{q}, 3\underline{p}, 6t)$. This α can be decomposed as $\alpha = \alpha_1 \alpha_3$, where $\alpha_1(\underline{q}, \underline{p}, t) := (\sqrt{\frac{2}{3}}\underline{q}, \sqrt{\frac{3}{2}}\underline{p}, t)$ and $\alpha_3(\underline{q}, \underline{p}, t) := (\sqrt{6}\underline{q}, \sqrt{6}\underline{p}, (\sqrt{6})^2 t) = (\sqrt{6}\underline{q}, \sqrt{6}\underline{p}, 6t)$. Further, α can be written as $\alpha(\underline{q}, \underline{p}, t) = (D_\alpha(\underline{q}, \underline{p}), 6t)$, where D_α is a dilative automorphism from $\mathbb{R}^{2d}$ to $\mathbb{R}^{2d}$ defined by $D_\alpha(\underline{q}, \underline{p}) := (2\underline{q}, 3\underline{p})$. Thus, we have $r_\alpha = 6$. Still using the same Γ as the one used in Example 5.3, we choose Γ_0 as the set $\Gamma_0 = \{(\underline{a}, c)\}$, where $a_j = 0$ or 1 for $1 \leq j \leq d$, $a_j = 0$, 1 or 2 for $d < j \leq 2d$ and $c = 0, \ 1/2, \ 1, \cdots, 5/2$. So the set A in Example 5.3 is a self-similar tile for the scaling system $(\mathbb{Z}^{2d}, D_\alpha)$ with dilated tile by D_α consisting of 6 original tiles. With this self similar tile in $\mathbb{R}^{2d}$, by Theorem 5.2 we obtain a self-similar stacked tile in $\mathbb{H}^d$:

$$T = \{(\underline{x}, t) \in \mathbb{H}^d : \underline{x} \in A, 0 \leq t - F(\underline{x}) < 1/2\}$$

with $F(\underline{x})$ constructed by

$$F(\underline{x}) = \sum_{m=1}^{\infty} \frac{1}{6^m} S([D_\alpha{}^m \underline{x}]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d})), < D_\alpha{}^m \underline{x} >_A),$$

where $[D_\alpha^m \underline{x}]_A \text{mod } (D_\alpha(\mathbb{Z}^{2d}))$ means $[2^m x_1]$ mod 2, $[2^m x_2]$ mod 2, ..., $[2^m x_n]$ mod 2 and $[3^m x_{n+1}]$ mod 3, $[3^m x_{n+2}]$ mod 3, ..., $[3^m x_{2n}]$ mod 3.

Generally speaking, whenever an automorphism α from $\mathbb{H}^{2d}$ to $\mathbb{H}^{2d}$ can be decomposed as $\alpha(\underline{q}, \underline{p}, t) = (D_\alpha(\underline{q}, \underline{p}), r_\alpha t)$ and there exists a self similar tile A in $\mathbb{R}^{2d}$ associated with D_α, then with this A, we can always construct a self similar tile in $\mathbb{H}^d$ associated with α.

The above functions F serve as scaling functions to generate MRAs for the space $L^2(\mathbb{H}^d)$. Since $F > 0$ has compact support, Proposition 3.6 shows that F is α-substantial. Therefore, F will generate MRAs for $L^2(\mathbb{H}^d)$ by Theorem 3.5. Theorem 4.6 guarantees the existence of Haar-like wavelet bases for the space $L^2(\mathbb{H}^d)$.

6. CONCLUSION

In this paper we are able to give the characterizations for a refinable function that is capable of generating an MRA in the space $L^2(G)$, where G is a locally compact group that does not have to be abelian. In deriving these characterizations, we did not use any information from the Plancherel side. In fact, for a general locally compact group, we may not be able to build the Fourier transform on it. However, in the case that the Fourier transform can be built up in the space $L^2(G)$ for some non-abelian locally compact groups, how can we characterize a refinable function to have a scaling function using the information from the Plancherel side? In particular, if G is a second countable, type I, unimodular locally compact group, do those results obtained by [**2**] in the space $L^2(\mathbb{R}^d)$ mentioned in the introduction still hold in the space $L^2(G)$? The authors intend to explore these questions in their future study.

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