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COMPACT OPEN SETS IN DUAL SPACES AND PROJECTIONS IN GROUP ALGEBRAS OF $[\mathrm{FC}]^-$ GROUPS

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Abstract. The structure of a compact open set in the dual of an $[\mathrm{FC}]^-$ group G, a locally compact group with relatively compact conjugacy classes, is given in terms of certain subsets which arise somewhat naturally. The support in the dual of a projection in $L^1(G)$ is a compact open set. Therefore, knowledge of the structure of such sets helps in identifying and constructing projections. We describe explicitly the compact open sets and construct projections for some illustrative examples.

Introduction

In the theory of harmonic analysis on a nonabelian compact group G, projections (selfadjoint idempotents) in $L^1(G)$ play an essential role and are constructed as appropriate coefficient functions of irreducible representations. In [5], an explicit construction of nontrivial projections in $L^1(G_{\mathrm{aff}})$, where G_{aff} is the group of affine transformations of $\mathbb{R}$, was given. In [8], and [13], techniques were developed by which projections could be explicitly constructed in $L^1(G)$ for special classes of noncompact locally compact G.

The key to identifying candidate groups for the existence of nontrivial projections is that the support, in the dual space $\widehat{G}$ of G, of a projection in $L^1(G)$ must be a compact open set in $\widehat{G}$. Open points in $\widehat{G}$ were studied in [2], [18] and [19] where connections were made to projections in $L^1(G)$ and $C^*(G)$, the group C^*-algebra. Projections in C^*-algebras of nilpotent groups were studied in [12] by exploiting knowledge of the possible compact open subsets of their duals. In [8] and [13], particular semidirect product groups were designed so that nontrivial compact open subsets of the dual exist.

For $f \in L^1(G)$, the identities which make it a projection ($f * f^* = f = f^*$) are sufficiently strong to suggest reconstruction formulas akin to the continuous wavelet transform. In [16], two-dimensional continuous wavelet transforms arising from irreducible representations of $\mathbb{H} \rtimes \mathbb{R}$, where $\mathbb{H}$ is the three dimensional Heisenberg group and $\mathbb{R}$ acts on $\mathbb{H}$ by automorphic dilations, were presented. This was based on techniques for constructing projections in $L^1(\mathbb{H} \rtimes \mathbb{R})$ developed in [13]. See [17] for a recent survey which includes a discussion of the interplay between compact open sets in $\widehat{G}$, projections in $L^1(G)$, and wavelet transforms.

Key words: Locally compact group; relatively compact conjugacy class; dual space; compact open set; group algebra; projection.

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The main purpose of this paper is to expand the class of groups G for which the structure of compact open sets in $\widehat{G}$ is known. The problem of identifying the compact open sets in the dual of an $[\mathrm{FC}]^-$ group is largely answered here. Recall that a (discrete) group G is an FC-group if every conjugacy class in G is finite. In analogy, a locally compact group G is said to be an $[\mathrm{FC}]^-$ group if each conjugacy class in G has a compact closure. This class of groups, which of course includes all locally compact groups with relatively compact commutator subgroup, has been thoroughly investigated in [9]. After collecting the necessary technical preliminaries in Section 1, the main results are stated and proven in Section 2. In Section 3 we comment on minimal compact open subsets of $\widehat{G}$ and determine all the compact open subsets for a class of illustrative examples. In the final section, constructions are given for projections in $L^1(G)$ for some examples of $[\mathrm{FC}]^-$ groups G.

1. Preliminaries

Let G be a locally compact group. The term representation of G will mean a weakly continuous unitary representation. The dual space of G is $\widehat{G}$, the set of equivalence classes of irreducible representations of G. For any representation ρ of G, the same symbol will be used for the associated $*$-representations of $L^1(G)$ and $C^*(G)$. Then $\ker\rho$ denotes its kernel in $C^*(G)$. An ideal in $C^*(G)$ is called primitive if it is the kernel of an irreducible representation and $\mathrm{Prim}(C^*(G))$ denotes the space of such ideals. It is endowed with the hull-kernel topology. The map $\rho \to \ker\rho$ is used to pull this topology back to $\widehat{G}$. Both [4] and [7] are good general references for these topics.

If H is a closed subgroup of G and π is a representation of G, then $\pi|_H$ denotes the restriction of π to H. If σ is a representation of H, then $\mathrm{ind}_H^G\,\sigma$ denotes the representation of G induced from σ (see [7]). If π is a representation of G and S is some set of representations, then $\pi \otimes S = \{\pi \otimes \rho : \rho \in S\}$.

Let L and H be locally compact groups and $G = L \times H$. If σ, respectively ρ, is a representation of L, resp. H, on the Hilbert space $\mathcal{H}_\sigma$, resp. $\mathcal{H}_\rho$, then $(\sigma \times \rho)(l,h) = \sigma(l) \otimes \rho(h)$, for $(l,h) \in G$, defines a representation of G on $\mathcal{H}_\sigma \otimes \mathcal{H}_\rho$. If $\pi \in \widehat{G}$ and if L is a Type I group, then there exist $\sigma \in \widehat{L}$ and $\rho \in \widehat{H}$ such that $\pi = \sigma \times \rho$ (see [4], 13.1.8).

If S and T are sets of unitary representations of G, then S is weakly contained in T ($S \prec T$) if $\cap_{\sigma\in S} \ker\sigma \supseteq \cap_{\tau\in T}\ker\tau$, and S and T are weakly equivalent ($S \sim T$) if $S \prec T$ and $T \prec S$. For a representation π of G, the support of π is the closed subset $\mathrm{supp}\,\pi = \{\rho \in \widehat{G} : \rho \prec \pi\}$ of $\widehat{G}$. Let N be a closed normal subgroup of G. The action of G on representations of N (in particular on $\widehat{N}$) is written as $(x,\tau) \to x\cdot\tau$, where $x\cdot\tau(n) = \tau(x^{-1}nx)$ for $x \in G$ and $n \in N$, and $G(\tau)$ and G_τ will denote the G-orbit under this action and the stability group of τ, respectively. Representations of G/N will frequently be viewed as representations of G, by composing with the quotient homomorphism $G \to G/N$. In particular, in this manner the dual space $\widehat{G/N}$

of G/N will be regarded as a closed subset of $\widehat{G}$. If K is a compact normal subgroup of G and π is a representation of G, let

$$\widehat{G}_{K,\pi} = \{\rho \in \widehat{G} : \rho|_K \sim \pi|_K\}.$$

Since $\widehat{K}$ is discrete, $\mathrm{supp}(\pi|_K)$ is open. Thus $\widehat{G}_{K,\pi}$ is an open subset of $\widehat{G}$ by continuity of restriction [6].

Let G be an $[\mathrm{FC}]^-$ group. Then it is known (see [9]) that (i) the set G^c of compact elements, that is, elements x of G such that the closed subgroup generated by x is compact, form a closed normal subgroup of G; (ii) G/G^c is abelian and $(G/G^c)^c = \{G^c\}$; (iii) each compact subset of G is contained in a compact, conjugation invariant set; and (iv) if G is compactly generated, then G^c is compact. Moreover, $\mathrm{Prim}(C^*(G))$ is a Hausdorff space (see [15] and [11]). It is worth mentioning that, conversely, if G is a connected locally compact group and $\widehat{G}$ is a Hausdorff space then G is an extension of a vector group by a compact connected group [1].

Example 1.1. A motivating family of examples is formed as follows. Let A be any abelian locally compact group and let its dual group $\widehat{A}$ act on $\mathbb{T} \times A$ by $\chi \cdot (z, a) = (\chi(a)z, a)$, for $(z, a) \in \mathbb{T} \times A$ and $\chi \in \widehat{A}$. Let $G = (\mathbb{T} \times A) \rtimes \widehat{A}$. Elements of G are written as (z, a, χ) with group product

$$(z_1, a_1, \chi_1)(z_2, a_2, \chi_2) = (\chi_1(a_2)z_1z_2, a_1a_2, \chi_1\chi_2).$$

If $A = \mathbb{R}$, then G is isomorphic to the reduced Heisenberg group obtained by taking the three dimensional Heisenberg group modulo a nontrivial discrete central subgroup. Note also that, by Pontryagin duality, there is complete symmetry in the roles of A and $\widehat{A}$. In particular, $(\mathbb{T} \times A) \rtimes \widehat{A}$ is isomorphic to $(\mathbb{T} \times \widehat{A}) \rtimes A$. The commutator subgroup of G is $\mathbb{T}$. Thus, G is an $[\mathrm{FC}]^-$ group.

Since $\mathbb{T}$ is a compact normal subgroup of G, $\mathbb{T} \subseteq G^c$ and

$$G^c/\mathbb{T} = (G/\mathbb{T})^c = A^c \times (\widehat{A})^c.$$

Thus, $G^c = \{(z, a, \chi) : z \in \mathbb{T}, a \in A^c, \chi \in (\widehat{A})^c\}$. It is known that $(\widehat{A})^c = \widehat{A/A_0}$, where A_0 is the connected component of the identity in A (see Theorem (24.17) of [10]).

2. Compact open sets in $\widehat{G}$

To a certain extent, sets of the form $\widehat{G}_{K,\pi}$ are the building blocks of arbitrary compact open sets in $\widehat{G}$ when G is an $[\mathrm{FC}]^-$ group. This is formulated precisely in two theorems.

Theorem 2.1. *Let G be an $[\mathrm{FC}]^-$ group and S a compact open subset of $\widehat{G}$. Then S is closed in $\widehat{G}$ and there exist a compact normal subgroup K of G which* *is open in G^c and finitely many elements $\pi_1, \dots, \pi_n$ of $\widehat{G}$ such that*

$$S = \bigcup_{j=1}^{n} \widehat{G}_{K,\pi_j}.$$

Let K and N be closed normal subgroups of an $[\mathrm{FC}]^-$ group G such that$K$ is compact, $K \subseteq N$ and N/K is connected and contained in the centre of G/K. Then, as we shall see in Lemma 2.4, the G-orbit closures $\overline{G(\tau)}$, $\tau \in \widehat{N}$, are minimal closed G-invariant subsets of $\widehat{N}$. We can therefore define an equivalence relation $\sim$ on $\widehat{N}$ by setting $\omega_1 \sim \omega_2$ if and only if $\overline{G(\omega_1)} = \overline{G(\omega_2)}$. Let $\widehat{N}/G$ denote the quotient space and equip it with the quotient topology. Moreover, for each $\rho \in \widehat{G}$, the support $\mathrm{supp}(\rho|_N)$ equals $\overline{G(\tau)}$ for some $\tau \in \widehat{N}$ (Lemma 2.5) and hence can be regarded as an element of $\widehat{N}/G$.

Theorem 2.2. *Let G be an $[\mathrm{FC}]^-$ group and let K be a compact normal subgroup of G such that K is open in G^c. Then, for $\pi \in \widehat{G}$, the set $\widehat{G}_{K,\pi}$ is compact if and only if there exist closed subgroups N and M of G with the following properties:*

(i) *$K \subseteq M \subseteq N$, N is open in G, N/K is a vector group and contained in the centre of G/K, and M/K is discrete.*

(ii) *There exists a homeomorphism between $\widehat{M/K}$ and the subset of $\widehat{N}/G$ consisting of all elements* $\mathrm{supp}(\rho|_N)$, *$\rho \in \widehat{G}_{K,\pi}$.*

Proof. (Of Theorem 2.1) We observe first that S is closed in $\widehat{G}$. To see this, consider the mapping $k : \widehat{G} \to \mathrm{Prim}(C^*(G))$ defined by $k(\sigma) = \ker \sigma$, and recall that $\widehat{G}$ carries the weak topology with respect to k. Hence $S = k^{-1}(k(S))$ as S is open. Moreover, $k(S)$ is compact and hence closed in $\mathrm{Prim}(C^*(G))$ since $\mathrm{Prim}(C^*(G))$ is a Hausdorff space [11]. Consequently, $S = k^{-1}(k(S))$ is closed in $\widehat{G}$.

Notice next that if $\pi \in S$, then $\pi \otimes \widehat{G/G^c} \subseteq S$. In fact, since G/G^c is abelian and compact-free, $\widehat{G/G^c}$ is connected and hence so is $\pi \otimes \widehat{G/G^c}$. Since S is open and closed in $\widehat{G}$, the statement follows.

Let $\mathcal{K}$ denote the collection of all compact normal subgroups of G which are open in G^c. Since every compact subset of G is contained in a G-invariant compact set and since a compact set consisting of compact elements generates a compact subgroup, it follows that $G^c = \bigcup\{K : K \in \mathcal{K}\}$.

We claim that given $\sigma \in S$, there exists $K \in \mathcal{K}$ such that $\widehat{G}_{K,\sigma} \subseteq S$. Towards a contradiction, assume that for each $K \in \mathcal{K}$ there exists $\sigma_K \in \widehat{G}$ such that $\sigma_K|_K \sim \sigma|_K$, but nevertheless $\sigma_K \notin S$. Then, by the above, $(\sigma_K \otimes \widehat{G/G^c}) \cap S = \emptyset$ for each $K \in \mathcal{K}$. Now

$$\sigma_K \otimes \widehat{G/G^c} \sim \sigma_K \otimes \mathrm{ind}_{G^c}^G 1_{G^c} = \mathrm{ind}_{G^c}^G(\sigma_K|_{G^c}),$$

so that, since S is open in $\widehat{G}$,

$$\mathrm{supp}\left(\mathrm{ind}_{G^c}^G(\sigma_K|_{G^c})\right) \cap S = \emptyset$$

for each $K \in \mathcal{K}$. On the other hand, since any compact subset of G^c is contained in K, for some $K \in \mathcal{K}$,

$$C_c(G^c) \subseteq \bigcup \{C^*(K) : K \in \mathcal{K}\}.$$

Thus, $\bigcup\{C^*(K) : K \in \mathcal{K}\}$ is dense in $C^*(G^c)$. Since $\sigma|_K \sim \sigma_K|_K$ for every $K \in \mathcal{K}$, it follows that

$$\sigma|_{G^c} \prec \{\sigma_K|_{G^c} : K \in \mathcal{K}\}.$$

This in turn implies, by continuity of inducing and since G/G^c is amenable,

$$\sigma \prec \mathrm{ind}_{G^c}^G(\sigma|_{G^c}) \prec \{\mathrm{ind}_{G^c}^G(\sigma_K|_{G^c}) : K \in \mathcal{K}\}.$$

Now, for each $K \in \mathcal{K}$, $\mathrm{ind}_{G^c}^G(\sigma_K|_{G^c})$ is weakly contained in $\widehat{G} \setminus S$. Since S is open, we conclude that $\sigma \in \widehat{G} \setminus S$.

This contradiction shows that given any $\sigma \in S$, there exists $K \in \mathcal{K}$ such that $\widehat{G}_{K,\sigma} \subseteq S$. Since the sets $\widehat{G}_{K,\sigma}$ are open in $\widehat{G}$ and S is compact, we find $K_1, \dots, K_m \in \mathcal{K}$ and $\sigma_1, \dots, \sigma_m \in \widehat{G}$ such that $S = \bigcup_{i=1}^m \widehat{G}_{K_i,\sigma_i}$.

Let now K denote the subgroup generated by $\bigcup_{i=1}^m K_i$. Then K is compact and open in G^c, and each K_i has finite index in K. Let K_0 be any one of the K_i and $\sigma = \sigma_i$. There exist finitely many $\tau_1 \ \dots, \tau_r \in \widehat{G}$ such that

$$\mathrm{ind}_{K_0}^G(\sigma|_{K_0}) \sim \{\mathrm{ind}_K^G(\tau_1|_K), \dots, \mathrm{ind}_K^G(\tau_r|_K)\}.$$

For any $\tau \in \widehat{G}$, we then have $\tau|_{K_0} \sim \sigma|_{K_0}$ if and only if $\tau|_K \sim \tau_i|_K$ for some $i \in \{1, \dots r, \}$. Thus

$$\widehat{G}_{K_0,\sigma} = \bigcup_{i=1}^r \widehat{G}_{K,\tau_i}.$$

Since $S = \bigcup_{i=1}^m \widehat{G}_{K_i,\sigma_i}$, it follows that there exist finitely many $\pi_1, \dots \pi_n \in \widehat{G}$ such that $S = \bigcup_{i=1}^n \widehat{G}_{K,\pi_i}$. This finishes the proof of Theorem 2.1. □

The proof of Theorem 2.2 is much more involved and requires a number of preliminary results.

Lemma 2.3. *Let N be an open normal subgroup of the locally compact group G and let $\tau \in \widehat{N}$ such that $\{\tau\}$ is open in $\widehat{N}$. Then the set*

$$S_\tau = \{\pi \in \widehat{G} : \pi|_N \not\prec \widehat{N} \setminus G(\tau)\}$$

is open and compact in $\widehat{G}$.

Proof. Since $\{\tau\}$ is open in $\widehat{N}$, there exists $f \in L^1(N)$ such that $\sigma(f) = 0$ for all $\sigma \in \widehat{N}$, $\sigma \neq \tau$, and $\tau(f) \neq 0$. Let g denote the trivial extension of f to all of G. If $\pi \in \widehat{G}$ is such that $\pi|_N \prec \widehat{N} \setminus G(\tau)$, then since $G(\tau)$ is open in $\widehat{N}$, $\pi(g) = \pi|_N(f) = 0$. On the other hand, if $\pi \in S_\tau$ then $G(\tau) \cap \mathrm{supp}(\pi|_N) \neq \emptyset$ and hence $\tau \in \mathrm{supp}(\pi|_N)$ since $\mathrm{supp}(\pi|_N)$ is G-invariant. As $\{\tau\}$ is open in $\widehat{N}$, $\tau \leq \pi|_N$ and therefore $\|\pi(g)\| = \|\pi|_N(f)\| \geq \|\tau(f)\|$. So

$$S_\tau = \{\pi \in \widehat{G} : \|\pi(g)\| \geq \|\tau(f)\|\},$$

which is a compact set. Clearly, S_τ is open in $\widehat{G}$ since $G(\tau)$ is open in $\widehat{N}$, and hence the set of all $\pi \in \widehat{G}$ such that $\pi|_N \prec \widehat{N} \setminus G(\tau)$ is closed in $\widehat{G}$. □

The proof of the following lemma is inspired by and similar to the proof of Lemma 3 of [11].

Lemma 2.4. *Let K and N be closed normal subgroups of the locally compact group G such that K is compact, $K \subseteq N$ and N/K is connected and contained in the centre of G/K. Given $\tau \in \widehat{N}$, there exist $\sigma \in \widehat{K}$ and a closed subgroup M of G such that $K \subseteq M \subseteq N$ and*

$$\overline{G_\sigma(\tau)} = \tau \otimes \widehat{N/M} \quad \text{and} \quad \overline{G(\tau)} = \bigcup_{a \in A} a \cdot \overline{G_\sigma(\tau)},$$

where A denotes a representative system for the left cosets of G_σ in G. Moreover, the sets $a \cdot \overline{G_\sigma(\tau)}$ are pairwise disjoint and open in $\overline{G(\tau)}$, and $\overline{G(\tau)}$ is a minimal closed G-invariant set.

Proof. Since K is compact and N/K is connected, each $\sigma \in \widehat{K}$ is N-invariant, and the sets $\widehat{N}_{K,\sigma} = \{\pi \in \widehat{N} : \pi|_K \sim \sigma\}$, $\sigma \in \widehat{K}$, are open and closed in $\widehat{N}$ and cover $\widehat{N}$. Moreover, since N/K is an abelian connected group, it is the direct product of a vector group and a compact group. So N is an extension of a compact group by a vector group and as such is type I (see [14], proof of Theorem 3.7). Theorem 2 of [11] now implies that $\widehat{N}_{K,\sigma} = \pi \otimes \widehat{N/K}$, for each $\pi \in \widehat{N}_{K,\sigma}$.

Since $\widehat{K}$ is discrete and N/K is connected, N-orbits in $\widehat{K}$ are simply singletons. Thus $\tau|_K \sim \sigma \in \widehat{K}$ for some $\sigma \in \widehat{K}$. If $x, a \in G$, then

$$x \cdot \tau \in \widehat{N}_{K,a\cdot\sigma} \iff (a^{-1}x) \cdot \tau|_K \sim \sigma \iff x \in aG_\sigma.$$

Thus $G(\tau) = \cup_{a\in A} a \cdot G_\sigma(\tau)$ and hence, since the sets $\widehat{N}_{K,a\cdot\sigma}$ are open and closed in $\widehat{N}$,

$$\overline{G(\tau)} = \bigcup_{a\in A} \left(\overline{G(\tau)} \cap \widehat{N}_{K,a\cdot\sigma} \right) = \bigcup_{a\in A} a \cdot \overline{G_\sigma(\tau)}.$$

Now, let $\Gamma = \{\gamma \in \widehat{N/K} : \tau \otimes \gamma = \tau\}$. Then Γ is a closed subgroup of $\widehat{N/K}$. If $x \in G_\sigma$, then $x \cdot \tau = \tau \otimes \gamma_x$ for some $\gamma_x \in \widehat{N/K}$. For each $x \in G_\sigma$, fix such a γ_x. Then, for $x, y \in G_\sigma$, since N/K is contained in the centre of G/K and hence every $\gamma \in \widehat{N/K}$ is G-invariant,

$$\tau \otimes \gamma_{yx} = (yx) \cdot \tau = y \cdot (x \cdot \tau) = y \cdot (\tau \otimes \gamma_x) = y \cdot \tau \otimes \gamma_x = \tau \otimes \gamma_y \gamma_x.$$

This shows that $\gamma_y \gamma_x \gamma_{yx}^{-1} \in \Gamma$. Thus the set $\cup_{x\in G_\sigma} \gamma_x \Gamma$ is a subgroup of $\widehat{N/K}$. So $\Delta = \overline{\cup_{x\in G_\sigma} \gamma_x \Gamma}$ is a closed subgroup of $\widehat{N/K}$ and therefore $\Delta = \widehat{N/M}$ for some closed subgroup M of G with $K \subseteq M \subseteq N$. Since $G_\sigma(\tau)\Gamma = G_\sigma(\tau)$ and $\tau \otimes \Delta$ is closed in $\widehat{N}$, it follows that $\overline{G_\sigma(\tau)} = \tau \otimes \Delta = \tau \otimes \widehat{N/M}$.

Finally, to show that $\overline{G(\tau)}$ is a minimal closed G-invariant set, let $\rho \in \overline{G(\tau)}$. Then $\rho = a\cdot(\tau\otimes\chi) = a\cdot\tau\otimes\chi$ for some $a \in A$ and $\chi \in \widehat{N/M}$. Since $a^{-1}\cdot\rho|_K \sim \sigma$, $\overline{G_\sigma(a^{-1}\cdot\rho)} = a^{-1}\cdot\rho\otimes\widehat{N/M}$ and this implies that $\tau = a^{-1}\cdot\rho\otimes\overline{\chi} \in \overline{G_\sigma(\rho)}$, whence $\overline{G(\tau)} \subseteq \overline{G(\rho)}$. □

Lemma 2.5. *Let G, N and K be as in Lemma 2.4. Let $\pi \in \widehat{G}$ and $\tau \in \widehat{N}$ such that $\pi|_N \sim G(\tau)$, and for τ let σ, M and A be as in Lemma 2.4. For each $\delta \in \widehat{M/K}$, choose $\gamma_\delta \in \widehat{N/K}$ with $\gamma_\delta|_M = \delta$. Then the map*

$$\delta \to G\left(\tau\otimes\gamma_\delta\widehat{N/M}\right) = \bigcup_{a\in A} a\cdot\overline{G_\sigma(\tau\otimes\gamma_\delta)}$$

is a bijection between $\widehat{M/K}$ and the collection of all sets $\mathrm{supp}(\rho|_N)$*, where $\rho \in \widehat{G}_{K,\pi}$.*

Proof. Let $\delta \in \widehat{M/K}$ and choose any $\rho \in \mathrm{supp}(\mathrm{ind}_N^G(\tau\otimes\gamma_\delta))$. Then

$$\rho|_K \prec \mathrm{ind}_N^G(\tau\otimes\gamma_\delta)|_K \sim G(\tau|_K) \sim G(\sigma) \sim \pi|_K$$

and hence $\rho \in \widehat{G}_{K,\pi}$. Similarly,

$$\rho|_N \prec G(\tau\otimes\gamma_\delta) = G(\tau)\otimes\gamma_\delta$$

and therefore, since $\overline{G(\tau)}$ is a minimal closed G-invariant set,

$$\mathrm{supp}(\rho|_N) = \overline{G(\tau)}\otimes\gamma_\delta = G(\tau\otimes\gamma_\sigma\widehat{N/M}).$$

If $\delta_1, \delta_2 \in \widehat{M/K}$ are such that

$$G\left(\tau\otimes\gamma_{\delta_1}\widehat{N/M}\right) = G\left(\tau\otimes\gamma_{\delta_2}\widehat{N/M}\right),$$

then, for some $a \in G$ and $\eta_1 \in \widehat{N/M}$, $a\cdot\tau = \tau\otimes\gamma_{\delta_1}\gamma_{\delta_2}^{-1}\eta_1$. This implies that $a \in G_\sigma$ and hence there exists $\eta_2 \in \widehat{N/M}$ such that $\tau = \tau\otimes\gamma_{\delta_1}\gamma_{\delta_2}^{-1}\eta_1\eta_2$. This shows that

$$\gamma = \gamma_{\delta_1}\gamma_{\delta_2}^{-1}\eta_1\eta_2 \in \Gamma \subseteq \widehat{N/M},$$

where Γ is as in the proof of Lemma 2.4. It follows that

$$\delta_1\delta_2^{-1} = (\gamma_{\delta_1}\gamma_{\delta_2}^{-1})|_M = \gamma|_M = 1_M,$$

so that $\delta_1 = \delta_2$. It remains to show that given $\rho \in \widehat{G}_{K,\pi}$, there exists $\delta \in \widehat{M/K}$ such that

$$\mathrm{supp}(\rho|_N) = G\left(\tau\otimes\gamma_\delta\widehat{N/M}\right).$$

There exists $\omega \in \widehat{N}$ such that $\omega|_K \sim \sigma$ and $\rho|_N \sim G(\omega)$. Then $\omega = \tau \otimes \eta$ for some $\eta \in \widehat{N/K}$ and

$$\begin{aligned} \mathrm{supp}(\rho|_N) &= \overline{G(\omega)} = \overline{G(\tau)} \otimes \eta = \tau \otimes \eta \widehat{N/M} \\ &= \tau \otimes \gamma_{\eta|_M} \left(\gamma_{\eta|_M}^{-1} \eta \right) \widehat{N/M} \\ &= \tau \otimes \gamma_{\eta|_M} \widehat{N/M} \end{aligned}$$

since $(\gamma_{\eta|_M}^{-1} \eta)|_M = 1_M$. □

Recall that the quotient space as $\widehat{N}/G$ of $\widehat{N}$ defined by the equivalence relation $\omega_1 \sim \omega_2$ if and only if $\overline{G(\omega_1)} = \overline{G(\omega_2)}$ for $\omega_1, \omega_2 \in \widehat{N}$, is equipped with the quotient topology.

Lemma 2.6. *Retain the assumptions and notation of Lemmas 2.4 and 2.5. Let $\pi \in \widehat{G}$ and $\tau \in \widehat{N}$ such that $\pi|_N \sim G(\tau)$, and let*

$$Q_\pi = \{\mathrm{supp}(\rho|_N) : \rho \in \widehat{G}_{K,\pi}\} \subseteq \widehat{N}/G.$$

Then the map $\delta \to G(\tau \otimes \gamma_\delta \widehat{N/M})$ of Lemma 2.5 is a homeomorphism from $\widehat{M/K}$ onto Q_π.

Proof. Let $(\delta_\alpha)_\alpha$ be a net in $\widehat{M/K}$ converging to some $\delta \in \widehat{M/K}$ and put $\gamma_\alpha = \gamma_{\delta_\alpha}$ and $\gamma = \gamma_\delta$. Then $\mathrm{ind}_M^N \delta_\alpha \to \mathrm{ind}_M^N \delta$, and since $\gamma \prec \mathrm{ind}_M^N \delta$, there exist $\eta_\alpha \in \widehat{N/M}$ such that $\gamma_\alpha \eta_\alpha \to \gamma$ in $\widehat{N/K}$. This implies $\tau \otimes \gamma_\alpha \eta_\alpha \to \tau \otimes \gamma$ in $\widehat{N}$ and therefore $\overline{G(\tau \otimes \gamma_\alpha \eta_\alpha)} \to \overline{G(\tau \otimes \gamma)}$ in Q_π.

Conversely, suppose that

$$\overline{G(\tau \otimes \gamma_\alpha \widehat{N/M})} \quad \to \quad \overline{G(\tau \otimes \gamma \widehat{N/M})}.$$

Then, since $\widehat{N}_{K,\tau}$ is open in $\widehat{N}$,

$$\begin{aligned} G_\sigma\left(\tau \otimes \gamma_\alpha \widehat{N/M}\right) &= G\left(\tau \otimes \gamma_\alpha \widehat{N/M}\right) \cap \widehat{N}_{K,\tau} \\ &\to G\left(\tau \otimes \gamma \widehat{N/M}\right) \cap \widehat{N}_{K,\tau} \\ &= G_\sigma\left(\tau \otimes \gamma \widehat{N/M}\right). \end{aligned}$$

Since $\overline{G_\sigma(\tau)} = \tau \otimes \widehat{N/M}$, restricting representations to M gives

$$\{\tau|_M \otimes \delta_\alpha\} = G_\sigma(\tau|_M \otimes \delta_\alpha) \to G_\sigma(\tau|_M \otimes \delta) = \{\tau|_M \otimes \delta\}.$$

Since M/K is a closed subgroup of a vector group, $M/K = D \times \mathbb{R}^d$, where $d \in \mathbb{N}_0$ and D is discrete. Let L be the pullback of $D \times \mathbb{Z}^d$ in G. It suffices to show that $(\delta_\alpha)_\alpha$ contains a subnet which converges pointwise on L/K to $\delta|_{L/K}$. Indeed, because characters of vector groups are linear, it then follows that the subnet converges to δ uniformly on compact subsets of M/K.

Now, since $\widehat{L/K}$ is compact, after passing to a subnet if necessary, we can assume that $\delta_\alpha|_L \to \eta$ in $\widehat{L/K}$ for some $\eta \in \widehat{L/K}$. Then

$$\tau|_L \otimes \delta|_L = \lim_\alpha (\tau|_L \otimes \delta_\alpha|_L) = \tau|_L \otimes \eta$$

and therefore $\tau|_L = \tau|_L \otimes (\delta|_L \cdot \eta^{-1})$. Since $\widehat{N}_{L,\tau} = \tau \otimes \widehat{N/L}$, it follows that $\tau = \tau \otimes \omega$ for some $\omega \in \widehat{N/K}$ extending $\delta|_L \cdot \eta^{-1}$. So $\omega \in \Gamma \subseteq \widehat{N/M}$ and consequently, as $L \subseteq M$, $\delta|_L = \eta = \lim_\alpha(\delta_\alpha|_L)$. This completes the proof. □

Corollary 2.7. *If $\widehat{G}_{K,\pi}$ is compact, then M/K is discrete.*

Proof. It follows from continuity of restricting representations that the map $\rho \to \mathrm{supp}(\rho|_N)$ from $\widehat{G}_{K,\pi}$ onto Q_π is continuous. Thus Q_π is compact, and hence so is $\widehat{M/K}$ by Lemma 2.6. Consequently, M/K is discrete. □

Lemma 2.8. *Let G be an $[\mathrm{FC}]^-$ group and let K and N be normal subgroups of G such that K is compact, N is open, $K \subseteq N$ and N/K is contained in the centre of G/K. Let $\pi \in \widehat{G}$ and $\sigma \in \widehat{N}$ be such that $\pi|_N \sim G(\sigma)$. Then $\widehat{G}_{K,\pi}$ is compact if either $\widehat{N}_{K,\sigma}$ or the subset*

$$S = \{\mathrm{supp}(\rho|_N) : \rho \in \widehat{G}_{K,\pi}\}$$

of $\widehat{N}/G$ is compact.

Proof. Let s denote the map $\tau \to \overline{G(\tau)}$ from $\widehat{N}$ onto $\widehat{N}/G$. Let $\tau \in \widehat{N}_{K,\sigma}$ and choose $\rho \in \widehat{G}$ such that $\rho|_N \sim G(\tau)$. Since $\overline{G(\tau)}$ is a minimal closed G-invariant set, ρ can be taken to be any element of $\mathrm{supp}(\mathrm{ind}_N^G \tau)$. Then

$$\rho|_K \sim G(\tau|_K) \sim G(\sigma|_K) \sim G(\sigma)|_K \sim \pi|_K,$$

and hence $\rho \in \widehat{G}_{K,\pi}$ and $s(\tau) \in S$. Conversely, if $\rho \in \widehat{G}_{K,\pi}$, then since N/K is contained in the centre of G/K, there exists $\chi \in \widehat{N/K}$ such that

$$\rho|_N \sim \pi|_N \otimes \chi \sim G(\sigma) \otimes \chi = G(\sigma \otimes \chi).$$

Since $\sigma \otimes \chi \in \widehat{N}_{K,\sigma}$, it follows that $\mathrm{supp}(\rho|_N) \in s(\widehat{N}_{K,\sigma})$. Thus we have seen that s maps $\widehat{N}_{K,\sigma}$ continuously onto S, and therefore compactness of $\widehat{N}_{K,\sigma}$ implies that S is compact.

It remains to show that if S is compact, then $\widehat{G}_{K,\pi}$ is compact. Since $\widehat{N}$ is locally compact and $s : \widehat{N} \to \widehat{N}/G$ is continuous and open, there exists a relatively compact open subset U of $\widehat{N}$ such that $U \cap \mathrm{supp}(\rho|_N) \neq \emptyset$ for each $\rho \in \widehat{G}_{K,\pi}$. As $\overline{U}$ is compact, we find $f \in L^1(N)$ such that $\|\omega(f)\| \geq \delta > 0$ for all $\omega \in \overline{U}$ and hence $\|\rho|_N(f)\| \geq \delta$ for each $\rho \in \widehat{G}_{K,\pi}$. Since N is open in G, we can view f as an element of $L^1(G)$, and we then have $\|\rho(f)\| = \|\rho|_N(f)\| \geq \delta$ for all $\rho \in \widehat{G}_{K,\pi}$. Thus $\widehat{G}_{K,\pi}$ is contained in a compact set and hence is compact since it is closed. □

In some of the preceding lemmas we have considered irreducible representations π of G such that $\pi|_N \sim G(\tau)$ for some $\tau \in \widehat{N}$. Such τ always exist when the normal subgroup N is second countable (see Lemma 1 of [11]). In passing we mention that the same is true in our context even though N need not be second countable.

Lemma 2.9. *Let K and N be normal subgroups of G such that K is compact, $K \subseteq N$, N is open in G and N/K is connected and second countable. Then, given $\pi \in \widehat{G}$, there exists $\tau \in \widehat{N}$ such that $\pi|_N \sim G(\tau)$.*

Proof. Choose $\sigma \in \mathrm{supp}(\pi|_K)$ and note that $G_\sigma \supseteq N$ since N/K is connected and $\widehat{K}$ is discrete. There exists $\rho \in \widehat{G_\sigma}$ such that $\pi \sim \mathrm{ind}_{G_\sigma}^G \rho$. Let C denote the kernel of σ in K. Then C is normal in G_σ and N/C is second countable since both K/C and N/K are second countable. As $\rho|_K \sim \sigma$, ρ can be viewed as an irreducible representation of G_σ/C. Then there exists $\tau \in \widehat{N/C} \subseteq \widehat{N}$ such that $\rho|_N \sim G_\sigma(\tau)$. Using that N is open in G and realizing $\mathrm{ind}_{G_\sigma}^G \rho$ in the Hilbert space $\ell^2(G/G_\sigma, \mathcal{H}_\rho)$, it is now straightforward to verify that

$$\left(\mathrm{ind}_{G_\sigma}^G \rho\right)|_N \sim \{x \cdot (\rho|_N) : x \in G\}.$$

This shows that $\pi|_N \sim G(\rho|_N) \sim G(\tau)$, as required. □

After all this preparation, a proof can be given to Theorem 2.2.

Proof. (Of Theorem 2.2) Note first that G/K is a Lie group. In fact, G^c/K is discrete and G/G^c is a compact-free abelian group, hence also a Lie group. We now define a normal subgroup N of G by

$$N = \{x \in G : xK \in (G/K)_0\}.$$

Then N is open in G. Observe next that N/K is compact-free because, since N/K is connected and G^c/K is discrete,

$$(N/K)^c = (G/K)^c \cap N/K = G^c/K \cap N/K = \{K\}.$$

Being a compact-free connected $[\mathrm{FC}]^-$ group, N/K is a vector group (see [9]). Moreover, for each $x \in G$, the set $[xK, N/K]$ consisting of all commutators $xyx^{-1}y^{-1}K$, $y \in N$, is connected and contained in the discrete group G^c/K because G/G^c is abelian. Consequently, N/K is contained in the centre of G/K.

Now suppose that $\widehat{G}_{K,\pi}$ is compact. By Lemma 2.9 there exists $\tau \in \widehat{N}$ such that $\pi|_N \sim G(\tau)$. By Lemma 2.4 there exist $\sigma \in \widehat{K}$ and a closed subgroup M of G such that the properties of Lemma 2.4 hold. Thus, by Lemma 2.6 and Corollary 2.7, conditions (i) and (ii) of Theorem 2.2 are satisfied.

Conversely, let $\pi \in \widehat{G}$ and let N and M be closed subgroups of G such that (i) and (ii) are satisfied. Then the collection of sets $\mathrm{supp}(\rho|_N)$, $\rho \in \widehat{G}_{K,\pi}$, is compact and hence so is $\widehat{G}_{K,\pi}$ by Lemma 2.8. □

3. Minimal compact open sets and Example 1.1

In this section we briefly comment on when a subset $\widehat{G}_{K,\pi}$ of $\widehat{G}$ is a minimal compact open set and then discuss groups as described in Example 1.1.

Proposition 3.1. *Let G be an* $[\mathrm{FC}]^-$ *group and $\pi \in \widehat{G}$. Let K be a compact normal subgroup of G such that K is open in G^c. Suppose that $\widehat{G}_{K,\pi}$ is compact. Then $\widehat{G}_{K,\pi}$ is a minimal compact open set if and only if π satisfies $\pi|_{G^c} \sim \mathrm{ind}_K^{G^c}(\pi|_K)$. In particular, $\widehat{G}_{K,\pi}$ is a minimal compact open subset of $\widehat{G}$ if $G^c = K$.*

Proof. Suppose first that $\mathrm{ind}_K^{G^c}(\pi|_K)$ is not weakly contained in $\pi|_{G^c}$. Let $\mathcal{L}$ denote the collection of all compact normal subgroups of G containing K. Then there exists $C \in \mathcal{L}$ such that $\mathrm{ind}_K^C(\pi|_K)$ is not weakly contained in $\pi|_C$ because otherwise, since every compact subset of G^c is contained in some $L \in \mathcal{L}$,

$$\pi|_L \succ \mathrm{ind}_K^L(\pi|_K) \sim \mathrm{ind}_K^{G^c}(\pi|_K)|_L$$

for each $L \in \mathcal{L}$ and hence $\mathrm{ind}_K^{G^c}(\pi|_K) \prec \pi|_{G^c}$. Choose $\sigma \in \widehat{C}$ with $\pi|_C \sim G(\sigma)$. Since $\widehat{C}$ is discrete, $G(\sigma)$ is a proper subset of the support of $\mathrm{ind}_K^C(\pi|_K)$ and $\widehat{G}_{C,\pi}$ is a proper open and closed subset of $\widehat{G}_{K,\pi} = \mathrm{supp}(\mathrm{ind}_K^G(\pi|_K))$.

Conversely, suppose that $\mathrm{ind}_K^{G^c}(\pi|_K) \prec \pi|_{G^c}$ and let C be any nonempty compact open subset of $\widehat{G}_{K,\pi}$. Then, for each $\rho \in C$, $\rho \otimes \widehat{G/G^c}$ is connected and $C \cap (\rho \otimes \widehat{G/G^c})$ is open and closed in $\rho \otimes \widehat{G/G^c}$. Thus $\rho \otimes \widehat{G/G^c} \subseteq C$. Since C is closed in $\widehat{G}$, $\widehat{G}_{K,\pi} \setminus C$ is also open and compact in $\widehat{G}_{K,\pi}$ and therefore we can assume that $\pi \in C$. Then, since $\pi|_{G^c} \succ \mathrm{ind}_K^{G^c}(\pi|_K)$ and G is amenable,

$$\begin{aligned} C &\sim \bigcup_{\rho \in C} \rho \otimes \widehat{G/G^c} \sim \{\mathrm{ind}_{G^c}^G(\rho|_{G^c}) : \rho \in C\} \\ &\succ \mathrm{ind}_{G^c}^G(\pi|_{G^c}) \succ \mathrm{ind}_{G^c}^G(\mathrm{ind}_K^{G^c}(\pi|_K)) \\ &\sim \mathrm{ind}_K^G(\pi|_K) \sim \widehat{G}_{K,\pi}. \end{aligned}$$

This shows that $C = \widehat{G}_{K,\pi}$, and hence $\widehat{G}_{K,\pi}$ is a minimal compact open subset of $\widehat{G}$. □

Corollary 3.2. *Suppose that G^c is contained in the centre of G and that $\widehat{G}_{K,\pi}$ is compact. Then $\widehat{G}_{K,\pi}$ is a minimal compact open subset of $\widehat{G}$ (if and) only if $K = G^c$.*

Proof. We have $\pi|_{G^c} \sim \tau$ for some $\tau \in \widehat{G^c}$ and therefore

$$\pi|_{G^c} \sim \mathrm{ind}_K^{G^c}(\pi|_K) \sim \mathrm{ind}_K^{G^c}(\tau|_K) \sim \tau \cdot \widehat{G^c/K},$$

which implies that $K = G^c$. □

Example 3.3. This example illustrates some of the complexity which can exist in general. For each $j \in \mathbb{N}$, let F_j be a finite group of order $|F_j| \geq 2$, and let $G = \prod'_{j \in \mathbb{N}} F_j$ be their restricted direct product. Then G is discrete, so $\widehat{G}$

is compact, and $G^c = G$. Let $\pi \in \widehat{G}$, and for each $n \in \mathbb{N}$, let $K_n = \prod_{j=1}^n F_j$ and $H_n = \prod'_{j\geq n+1} F_j$. Then K_n is Type I since it is finite. Thus $\pi = \sigma_n \times \tau_n$, for some $\sigma_n \in \widehat{K_n}$ and $\tau_n \in \widehat{H_n}$. Hence $\pi|_{K_n}$ is a multiple of σ_n and $\widehat{G}_{K_n,\pi} = \{\sigma_n\} \times \widehat{H_n}$. Notice that $\sigma_{n+1} = \sigma_n \times \omega_n$, for some $\omega_n \in \widehat{F_{n+1}}$ and $|\widehat{F_{n+1}}| \geq 2$. Thus, $\widehat{G}_{K_{n+1},\pi} \subsetneq \widehat{G}_{K_n,\pi}$. If $\rho \in \widehat{G}$ and $\rho \notin \overline{\{\pi\}}$, then there exists n such that $\rho|_{K_n} \not\sim \pi|_{K_n}$. Therefore, $\bigcap_{n\in\mathbb{N}} \widehat{G}_{K_n,\pi} = \overline{\{\pi\}}$ which is not open in $\widehat{G}$.

If now K is an arbitrary finite normal subgroup of G, then $K \subseteq K_n$ for some n, so $\widehat{G}_{K_n,\pi} \subseteq \widehat{G}_{K,\pi}$. Thus, Theorem 2.1 and the above shows that $\widehat{G}_{K,\pi}$ does not contain any minimal closed and open subset.

A characterization of when sets of the form $\widehat{G}_{K,\pi}$ are compact can be made more explicit in a form which is easily checked in the case of the kinds of groups described in Example 1.1.

Theorem 3.4. *Let A be a compactly generated locally compact abelian group and let $G = (\mathbb{T} \times A) \rtimes \widehat{A}$. Let $\pi \in \widehat{G}$ and let K be a compact normal subgroup of G which is open in G^c.*

(i) *If $\pi|_{\mathbb{T}} \not\sim 1_{\mathbb{T}}$, then the set $\widehat{G}_{K,\pi}$ is compact.*

(ii) *If $\pi|_{\mathbb{T}} \sim 1_{\mathbb{T}}$, then $\widehat{G}_{K,\pi}$ is compact if and only if A is finite.*

Proof. (i) By the structure theorem for compactly generated locally compact abelian groups (see [10], Theorem (9.8)), $A = \mathbb{R}^n \times \mathbb{Z}^m \times B$, where B is compact and $n, m \in \mathbb{N}_0$. We identify $\mathbb{R}^n, \widehat{\mathbb{R}^n}, \mathbb{Z}^m, \mathbb{T}^m = \widehat{\mathbb{Z}^m}$ and $B, \widehat{B}$ with the corresponding subgroups of $G/\mathbb{T}$.

Since K is open in G^c, $K \supseteq \mathbb{T}$ and hence

$$K/\mathbb{T} \subseteq G^c/\mathbb{T} = (G/\mathbb{T})^c = A^c \times \widehat{A/A_0} = B \times \mathbb{T}^m \times \widehat{B}^t.$$

Moreover, $K/\mathbb{T}$ contains $\mathbb{T}^m$ as well as a subgroup of finite index in B. Also note that if K is a subgroup of finite index in a normal subgroup C of G, then as shown in the proof of Theorem 2.1, $\widehat{G}_{K,\pi}$ is a finite union of sets $\widehat{G}_{C,\rho}$, $\rho \in \widehat{G}$. Therefore we can assume that

$$K/\mathbb{T} = B \times \mathbb{T}^m \times F,$$

where F is a finite subgroup of $\widehat{B}$. By hypothesis, there exists $q \in \mathbb{Z}$, $q \neq 0$, such that $\pi(z) = z^q$ for all $z \in \mathbb{T}$. Now, for any group H, let H_q denote the subgroup of H consisting of all elements h with $h^q = e$. Thus $\mathbb{T}_q$ is the kernel of $\pi|_{\mathbb{T}}$. Let

$$L = \{g \in G : [g, G] \subseteq \mathbb{T}_q\} = \{g \in G : gyg^{-1}y^{-1} \in \mathbb{T}_q \text{ for all } y \in G\}$$

, that is, $L/\mathbb{T}_q$ is the centre of $G/\mathbb{T}_q$. Since $\mathbb{T}$ is contained in the centre of G and $G/\mathbb{T}$ is abelian, there exists a character λ of L such that $\pi \sim \operatorname{ind}_L^G \lambda$ ([3], Lemma 3.1). It is now straightforward to verify that

$$L = \{g = (z, a, \chi) \in G : \overline{\chi(a')}\chi'(a) \in \mathbb{T}_q \text{ for all } a' \in A \text{ and } \chi' \in \widehat{A}\}.$$

To describe L more explicitly, let $g = (z, a, \chi) \in L$ and $a = (x, y, t), x \in \mathbb{R}^n, y \in \mathbb{Z}^m$ and $t \in B$. Taking for a' the identity of A, it follows that $\chi'(qx) = \chi'(x)^q = 1$ for all $\chi' \in \widehat{\mathbb{R}^n}$ and hence $x = 0$. Similarly, we get $\chi'(qy) = \chi'(y)^q = 1$ for all $\chi' \in \widehat{\mathbb{Z}^m}$, whence $y = 0$, and $\chi'(t^q) = 1$ for all $\chi' \in \widehat{B}$, which implies $t \in B_q$. Thus we have seen that $a \in B_q$. Furthermore, with $\chi' = 1_A$, we have $\chi(qa') = \chi(a')^q = 1$ for all $a' \in \mathbb{R}^n$ and hence $\chi|_{\mathbb{R}^n} = 1_{\mathbb{R}^n}$. Consequently, $\chi \in \widehat{\mathbb{Z}^m \times B} = \mathbb{T}^m \times \widehat{B}$, and then arguments as above show that actually $\chi \in \mathbb{T}^m_q \times \widehat{B}_q$. Summing up, we have

$$L/\mathbb{T} \subseteq B_q \times \mathbb{T}^m_q \times \widehat{B}_q,$$

the converse inclusion being obvious.

Define a normal subgroup N of G by $N \supseteq K$ and $N/K = \mathbb{R}^n \times \widehat{\mathbb{R}^n}$. Then N is open in G and N/K is a vector group. We determine the set

$$S = \{\mathrm{supp}(\rho|_N) : \rho \in \widehat{G}_{K,\pi}\}.$$

Since $\pi \sim \mathrm{ind}_L^G \lambda$ and G/K is abelian, an element ρ of $\widehat{G}$ belongs to $\widehat{G}_{K,\pi}$ if and only if there exists $\chi \in \widehat{N/K}$ such that

$$\rho|_N \sim \chi \otimes \pi|_N \sim \chi \otimes \mathrm{ind}^N_{L\cap N}(\lambda|_{L\cap N}) = \mathrm{ind}^N_{L\cap N}(\chi|_{L\cap N} \otimes \lambda|_{L\cap N}).$$

Now, by definition of N and the description of L,

$$(L \cap N)/\mathbb{T} = L/\mathbb{T} \cap N/\mathbb{T} = \left(B_q \times \mathbb{T}^m_q \times \widehat{B}_q\right) \cap \left(\mathbb{R}^n \times \widehat{\mathbb{R}^n}\right),$$

which is the trivial subgroup of $G/\mathbb{T}$. So $L \cap N = \mathbb{T}$ and therefore

$$\rho|_N \sim \mathrm{ind}^N_{\mathbb{T}}(\chi|_{\mathbb{T}} \otimes \lambda|_{\mathbb{T}}) = \mathrm{ind}^N_{\mathbb{T}}(\lambda|_{\mathbb{T}})$$

for each $\rho \in \widehat{G}_{K,\pi}$. This shows that S is a singleton, $\{\mathrm{supp}(\mathrm{ind}^N_{\mathbb{T}}(\lambda|_{\mathbb{T}}))\}$, and hence $\widehat{G}_{K,\pi}$ is compact by Lemma 2.8. The hypotheses of Theorem 2.2 are now satisfied when taking $M = K$.

(ii) If $\pi \in \widehat{G/\mathbb{T}}$, then $\widehat{G}_{K,\pi} = \pi \otimes \widehat{G/K}$ and since

$$G/K = \mathbb{R}^n \times \mathbb{Z}^m \times \widehat{\mathbb{R}^n} \times \widehat{B}/F,$$

compactness of $\widehat{G}_{K,\pi}$ is equivalent to $n = m = 0$ and finiteness of $\widehat{B}$, which means that A is finite. □

4. Examples of projections in $L^1(G)$

If f is a nonzero projection in $C^*(G)$, then associated with f is the nonempty compact open set

$$S(f) = \{\pi \in \widehat{G} : \pi(f) \neq 0\}$$

of $\widehat{G}$, the *support of* f (see [4], (3.3.2), (3.3.7)). If G is abelian, then $f \to S(f)$ is a bijective correspondence between nonzero projections in $C^*(G)$ and nonempty compact open subsets of the dual group $\widehat{G}$. For nonabelian G, the map $f \to S(f)$ will no longer be one-to-one. Also, even for 2-step solvable discrete groups G, a compact open subset of $\widehat{G}$ need not be the support set of some projection (see [12], Example 2 and Proposition).

The partial ordering on the set of projections in $L^1(G)$ is given by $g \leq f$ if $g = g * f$ (and hence also $g = f * g$). Thus a nonzero projection f is said to be minimal if $g \leq f$ implies $g = 0$ or $g = f$.

Proposition 4.1. *Let G be an* $[\mathrm{FC}]^-$ *group such that G^c is open in G and let S be a compact open subset of $\widehat{G}$. Then there exists a projection f in $L^1(G)$ such that $S(f) = S$. The conclusion in particular holds if G/G^c is totally disconnected.*

Proof. Note first that if G/G^c is totally disconnected, then G/G^c is discrete since G/G^c is compact-free. Therefore, it suffices to prove the first statement of the lemma.

By Theorem 2.1, there exists a compact normal subgroup K of G which is open in G^c, and hence open in G, such that S is a union of finitely many sets of the form $\widehat{G}_{K,\pi}$, $\pi \in \widehat{G}$. We fix π and show there is a projection $f \in L^1(G)$ with $S(f) = \widehat{G}_{K,\pi}$. Choose $\sigma \in \widehat{K}$ such that $\overline{\pi}|_K \geq \sigma$ and a unit vector ξ in the space of σ. Let d_σ denote the dimension of σ and put $f(x) = d_\sigma \langle \sigma(x)\xi, \xi\rangle$ for $x \in K$ and $f(x) = 0$ for $x \in G \setminus K$. Then $f^* = f$ and the orthogonality relations for irreducible representations of compact groups show that f is an idempotent. In fact, $(f * f)(x) = f(x) = 0$ for $x \in G \setminus K$ and, for $x \in K$,

$$
\begin{aligned}
(f * f)(x) &= d_\sigma^2 \int_K \langle \sigma(xy)\xi, \xi\rangle \langle \sigma(y^{-1})\xi, \xi\rangle dy \\
&= d_\sigma^2 \int_K \langle \sigma(y)\xi, \sigma(x^{-1})\xi\rangle \overline{\langle \sigma(y)\xi, \xi\rangle} dy \\
&= d_\sigma^2 \frac{1}{d_\sigma} \overline{\langle \sigma(x^{-1})\xi, \xi\rangle} \\
&= d_\sigma \langle \sigma(x)\xi, \xi\rangle = f(x).
\end{aligned}
$$

Using the orthogonality relations again, it is easily verified that the projection f has the property that $S(f) = \widehat{G}_{K,\pi}$. Finally, let $S = \cup_{j=1}^n \widehat{G}_{K,\pi_j}$, let f_j be as above associated with $\widehat{G}_{K,\pi_j}$, $1 \leq j \leq n$ and put $f = \sum_{j=1}^n f_j$. Then, for any $\pi \in \widehat{G}$, $\pi(f) = 0$ whenever $\pi \notin S$ and $\pi(f) = \pi(f_k)$ if $\pi \in \widehat{G}_{K,\pi_k}$. Thus $S(f) = S$ and $\pi(f * f) = \pi(f)$ for all $\pi \in \widehat{G}$, which implies that f is a projection. $\square$

Example 4.2. Consider the reduced Heisenberg group $H = \{(z, t, x) : z \in \mathbb{T}, t, x \in \mathbb{R}\}$ with the product $(z_1, t_1, x_1)(z_2, t_2, x_2) = (e^{2\pi i t_2 x_1} z_1 z_2, t_1 + t_2, x_1 + x_2)$. This is a special case of Example 1.1 Then $H^c = \mathbb{T} \times \{0\} \times \{0\}$ and $H/H^c = \mathbb{R}^2$. Projections in $L^1(H)$ were constructed in [12] using the orthogonality relations for integrable representations of unimodular groups from [4].

The next example is a closed subgroup of H from Example 4.2 where there are no longer open points in the dual, so the integrable representation technique of [12] no longer applies. Nevertheless, there are still compact open sets in the dual and a modification of the construction gives projections in $L^1(G)$.

Example 4.3. Let G be the semidirect product $G = (\mathbb{T} \times \mathbb{R}) \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{T} \times \mathbb{R}$ by

$$((z,t),n) \to n \cdot (z,t) = (e^{2\pi itn}z, t), \quad z \in \mathbb{T}, t \in \mathbb{R}, n \in \mathbb{Z}.$$

We identify $\mathbb{T}$ with $\mathbb{T} \times \{0\} \times \{0\}$, the centre of G, $\mathbb{T} \times \mathbb{R}$ with $N = \mathbb{T} \times \mathbb{R} \times \{0\}$ and $\mathbb{Z}$ with $\{1\} \times \{0\} \times \mathbb{Z}$. For $m \in \mathbb{Z} = \widehat{\mathbb{T}}$ and $t \in \mathbb{R} = \widehat{\mathbb{R}}$ let $\chi_{m,t}$ be the character of $\mathbb{T} \times \mathbb{R}$ given by $\chi_{m,t}(z,s) = z^m e^{2\pi ist}$. Then the action of $\mathbb{Z}$ on $\widehat{N}$ is given by $n \cdot \chi_{m,t} = \chi_{m,t+nm}$, so that $G(\chi_{m,t}) = \{m\} \times \{t + nm : n \in \mathbb{Z}\}$. Since the orbits in $\widehat{N}$ are closed and discrete, G is type I. For $m \in \mathbb{Z}$, put $\widehat{G}_m = \{\pi \in \widehat{G} : \pi|_{\mathbb{T}} \sim m\}$. Then $\widehat{G}_0 = \widehat{G/\mathbb{T}} = \widehat{\mathbb{Z} \times \mathbb{R}} = \mathbb{T} \times \mathbb{R}$ does not contain any nonempty compact open subset. However, for $m \neq 0$, $\widehat{G}_m$ is a minimal compact open subset of $\widehat{G}$. In fact, since the stability group of $\chi_{m,t}$ equals N and $G(\chi_{m,t}) = \{m\} \times (t + \mathbb{Z}m)$, the map $t \to \mathrm{ind}_N^G \chi_{m,t}$ is a continuous map from, $[0,m]$ onto $\widehat{G}_m$, so that $\widehat{G}_m$ is compact and connected.

We fix $m \in \mathbb{Z}$, $m \neq 0$, and employ the concept of a projection generating function, as introduced in [8] and used in [8] and [13], to construct a projection $f \in L^1(G)$ such that $S(f) = \widehat{G}_m$. Let $U = \{m\} \times \mathbb{R} \subseteq \widehat{N}$ and let π_U denote the unitary representation of G on $L^2(U) \subseteq L^2(\widehat{N})$ defined by

$$\pi_U(a,n)\xi(\chi) = \chi(a)\xi(n \cdot \chi), \quad \xi \in L^2(U),$$

for $a \in N, n \in \mathbb{Z}$ and $\chi \in U$. Moreover, for $\xi \in L^2(U)$, define a function f_ξ on G by

$$f_\xi(a,n) = \langle \xi, \pi_U(a,n)\xi \rangle = (\xi \overline{n \cdot \xi})^\vee(a),$$

for $(a,n) \in G$. By Theorem 1.2 of [13], f_ξ is a projection with $S(f_\xi) = \widehat{G}_m$ provided that ξ is a projection generating function associated with U, that is, a measurable function on $\widehat{N}$ with the following properties:

(i) $\xi(\chi) = 0$ for all $\chi \in \widehat{N} \setminus U$;
(ii) For all $n \in \mathbb{Z}$, $\xi \overline{n \cdot \xi} = \widehat{h_n}$ for some $h_n \in L^1(N)$;
(iii) $f_\xi \in L^1(G)$;
(iv) $\sum_{n \in \mathbb{Z}} |\xi(n \cdot \chi)|^2 = 1$ for every $\xi \in U$.

Here is an example of such a projection generating function ξ. Let $\varphi \in C_c^\infty(\mathbb{R})$ be such that $\operatorname{supp}\varphi \subseteq [-m/2, 3m/2]$, $\varphi \geq 0$ and $\varphi(t) > \delta$, for all $t \in [0,m]$, where $\delta > 0$ is fixed. For $t \in \mathbb{R}$, let

$$\beta(t) = \left[\sum_{n \in \mathbb{Z}} \varphi(t+nm)^2\right]^{-1/2}.$$

Then $\beta \in C^\infty(\mathbb{R})$, $\beta(t) \geq \delta$ and $\beta(t+m) = \beta(t)$, for all $t \in \mathbb{R}$. Thus, φ/β is a Schwartz function on $\mathbb{R}$ satisfying

$$\sum_{n \in \mathbb{Z}} |(\varphi/\beta)(t+nm)|^2 = 1, \tag{1}$$

for all $t \in \mathbb{R}$. Let w be a Schwartz function on $\mathbb{R}$ such that $\widehat{w} = \varphi/\beta$.

Define ξ on $\widehat{N}$ by $\xi(\chi_{m,t}) = \varphi(t)/\beta(t)$ and $\xi(\chi_{m,t}) = 0$ if $k \neq m$. Define $g : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$ by $g(z,s) = z^{-m}w(s)$. Then $\xi = \widehat{g}$.

Property (i) above holds by the definition of ξ and (1) implies property (iv). For each $n \in \mathbb{Z}$, let $\tau_n g(z,t) = g((-n) \cdot (z,t)) = g(e^{-2\pi i t n} z, t)$. Then $g * (\tau_n g)^* \in L^1(N)$ and $(g * (\tau_n g)^*)\widehat{\ } = \xi\,\overline{n \cdot \xi}$, for each $n \in \mathbb{Z}$. Thus property (ii) holds. Observe next that, for each $t \in \mathbb{R}$, $\widehat{g}(\chi_{k,t}) = 0$ for $k \neq m$ and

$$(g * (\tau_n g)^*)\widehat{\ }(\chi_{m,t}) = \xi\,\overline{n \cdot \xi}(\chi_{m,t}) = \widehat{w}(t)\widehat{w}(t + nm) = 0,$$

for $|n| \geq 2$. So $g * (\tau_n g)^* = 0$ for all $|n| \geq 2$. Now with $f_\xi(a,n) = (\xi\,\overline{n \cdot \xi})^\vee(a)$ for $a \in N$ and $n \in \mathbb{Z}$. Then

$$\begin{aligned}
\int_G |f_\xi(x)| dx &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \int_{\mathbb{R}} |f_\xi(z,s,n)|\, dz ds \\
&= \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \int_{\mathbb{R}} \left|(\xi \overline{n \cdot \xi})^\vee (z,s)\right| dz ds \\
&= \sum_{n \in \mathbb{Z}} \int_{\mathbb{T}} \int_{\mathbb{R}} |(g * (\tau_n g)^*)(z,s)|\, dz ds \\
&= \sum_{n=-1}^{n=1} \|g * (\tau_n g)^*\|_1 \leq \|g\|_1 \sum_{n=-1}^{n=1} \|\tau_n g\|_1 < \infty,
\end{aligned}$$

which proves (iii).

Thus, f_ξ is a projection in $L^1(G)$ with $\operatorname{supp} f_\xi = \widehat{G}_m$.

Remark 4.4. The reasoning of Proposition 3.2 of [13] shows that the projection f_ξ constructed in Example 4.3 is minimal. For each $\chi \in \widehat{N}$, let $\pi_\chi = \operatorname{ind}_N^G \chi$. Then $\{\pi_\chi : \chi \in \widehat{N}\}$ is a faithful family of representations on $L^1(G)$. Suppose $f \in L^1(G)$ is a projection such that $0 \neq f \leq f_\xi$. Thus, $\operatorname{supp} f \subseteq \operatorname{supp} f_\xi$. But $\operatorname{supp} f_\xi = \widehat{G}_m$, which is connected. So $\operatorname{supp} f = \operatorname{supp} f_\xi$. Then, as in Proposition 3.2 of [13], $\pi_\chi(f_\xi)$ is a rank one projection for each χ of the form $\chi_{m,t}$, but $\pi_\chi(f_\xi) = 0$ if χ is not of this form. This implies $\pi_\chi(f) = \pi_\chi(f_\xi)$, for all $\chi \in \widehat{N}$. Thus $f = f_\xi$. That is, f_ξ is minimal.

5. Concluding remarks

Theorems 2.1 and 2.2 provide a complete characterization of the compact open sets in $\widehat{G}$ for any $[\mathrm{FC}]^-$ group G. The characterization is made very explicit in Theorem 3.4 for groups of the form $G = (\mathbb{T} \times A) \rtimes \widehat{A}$, where A is compactly generated and abelian. In some particular examples, projections in $L^1(G)$ are constructed with support a given compact open set. However, the following questions remain.

Question 1. Is every compact open set in $\widehat{G}$ the support set of a projection in $L^1(G)$ if G is an $[\mathrm{FC}]^-$ group? In [12], it is shown that, for $G = \mathbb{Z} \rtimes \mathbb{Z}_2$ with the nontrivial action of $\mathbb{Z}_2$, there are compact open subsets of $\widehat{G}$ which are not the support sets of projections. However, each nonempty compact open set contains the support set of some nonzero projection. Note that $\mathbb{Z} \rtimes \mathbb{Z}_2$ is not an $[\mathrm{FC}]^-$ group.

Question 2. Is it true that, for any locally compact group G, if S is a nonempty compact open subset of $\widehat{G}$, then there exists a nonzero projection f in $L^1(G)$ with $S(f) \subseteq S$?

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