Subspaces of $L^2(\mathbb{R}^n)$ Invariant Under Shifts by a Crystal Group

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January 8, 2026

Abstract

For a crystal group Γ in dimension n, a closed subspace $\mathcal{V}$ of $L^2(\mathbb{R}^n)$ is called Γ–shift invariant if, for every $f \in \mathcal{V}$, the shifts of f by every element of Γ also belong to $\mathcal{V}$. The main purpose of this paper is to provide a characterization of the Γ–shift invariant closed subspaces of $L^2(\mathbb{R}^n)$.

1 Introduction

Let Γ be a crystal group in dimension n with point group Π and associated lattice $\mathcal{L}$. Detailed definitions are given in Section 3. Elements of Γ are written $[x, M]$, where M is an orthogonal matrix that belongs to Π and x is a vector in $\mathbb{R}^n$. For a function f on $\mathbb{R}^n$, the *shift* of f by $[x, M]$ is $\pi[x, M]f$ given by

$$\pi[x, M]f(z) = f\left(M^{-1}z - x\right),$$

for $z \in \mathbb{R}^n$. When these shifts are applied to functions in the Hilbert space of all square-integrable functions, $L^2(\mathbb{R}^n)$, they constitute a unitary representation of Γ, which we call the natural representation. A closed subspace $\mathcal{V}$ of $L^2(\mathbb{R}^n)$ is called π-invariant, or Γ–shift invariant, if $\pi[x, M]f \in \mathcal{V}$, for every $f \in \mathcal{V}$ and every $[x, M] \in \Gamma$. Our purpose is to provide a characterization of the Γ–shift invariant closed subspaces of $L^2(\mathbb{R}^n)$.

The term shift-invariant closed subspace means a closed subspace $\mathcal{V}$ of $L^2(\mathbb{R}^n)$ such that $T_k f \in \mathcal{V}$, for all $f \in \mathcal{V}$ and all $k \in \mathbb{Z}^n$, where $T_k f(z) = f(z - k)$, for $z \in \mathbb{R}^n$. In every dimension, there is one crystal group that is abelian and it is the set of shifts by lattice points; so shift-invariance is Γ–shift invariance when Γ is isomorphic to $\mathbb{Z}^n$. Understanding shift-invariant subspaces became more important with the rise of wavelet analysis (see [9] for a general introduction to the classical theory of wavelets) as the central subspace in a multiresolution analysis is shift-invariant; see also [18] for more on the role of shift-invariant subspaces in wavelet theory. A characterization of shift-invariant subspaces was known in 1964 by Helson [14] and was reformulated in 2000 by Bownik [5] where he applied it to wavelet theory.

The theory of wavelets has many generalizations and variations. One generalization, wavelets with composite dilations, was introduced in [12] and [13]. The term *composite dilations* refers to the addition of another set $\mathcal{B}$ of matrices that are moving functions, besides the shifts by vectors in a lattice and the powers of a single dilating matrix. Often, $\mathcal{B}$ is a finite group of measure preserving matrices. Blanchard developed the theory in this case in [3]; see also [4] where the focus was on developing Haar-type wavelet systems when $\mathcal{B}$ leaves the shift lattice invariant and the semi-direct product of $\mathcal{B}$ with the lattice is actually a crystal group, necessarily symmorphic. In [16] MacArthur and one of the current authors introduced the concept of a multiresolution analysis where the shifts came from an arbitrary crystal group and cast the theory in the context of the abstract approach of [2]. Independently, González and Moure [11] also formulated the theory for shifts by a crystal group. In the definition of a multiresolution analysis or a generalized multiresolution analysis for either composite dilations, with a finite $\mathcal{B}$, or shifts by a crystal group, the central subspace in the multiresolution analysis is left invariant under shifts by a crystal group Γ.

Another reason to be concerned about the nature of Γ–shift invariant closed subspaces is the emergence of topological quantum chemistry (see [8] and [7], for example) where band representations play a significant role. A band representation of a crystal group Γ is the restriction of the natural representation π to a closed Γ–shift invariant subspace generated by a single function in $L^2(\mathbb{R}^n)$. A readable review of band representations can be found in [1].

Our characterization of Γ–shift invariant closed subspaces of $L^2(\mathbb{R}^n)$ is stated in Theorem 6.3 with a variation given in Corollary 6.7. Imprecisely stated, there is a one-to-one correspondence between Γ–shift invariant closed subspaces of $L^2(\mathbb{R}^n)$ and certain maps (the range functions) from an open subset of $\mathbb{R}^n$ and the closed subspaces of $\ell^2(\mathcal{L}^*)$, where $\mathcal{L}^*$ is the dual lattice of $\mathcal{L}$. If $\Gamma = \mathbb{Z}^n$, then the characterization in Theorem 6.3 reduces to the characterization of shift-invariant subspaces given in Proposition 1.5 in [5]. Our approach draws on the presentation of range functions by Bownik and Ross in [6], to the extent that Γ–shift invariance implies invariance under shifts by the lattice $\mathcal{L}$. Perhaps the feature of our characterization that is most interesting arises when Γ is non-symmorphic; that is Γ is not a semi-direct product of $\mathcal{L}$ with the point group Π. We view π as a unitary representation of Γ and our strategy is to use unitary maps to transform π into an equivalent representation that can be analyzed using range function methods.

We gather together preliminary facts and known results in Section 2 while Section 3 is devoted to organizing the properties of crystal groups that are needed. The unitary maps that will be used to transform π are introduced in Section 4 and applied in the next section. In the final section, we prove our main theorem, Theorem 6.3. We also present an example of a non-symmorphic wallpaper group Γ and a Γ–shift invariant closed subspace of $L^2(\mathbb{R}^2)$ where the impact of the non-symmorphic nature of this Γ is illustrated.

2 Preliminaries

Let n be a positive integer and let $\mathbb{R}^n$ denote Euclidean space with $x \in \mathbb{R}^n$ being considered as a column vector. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^n)$ denotes the standard Lebesgue space with respect to Lebesgue measure on $\mathbb{R}^n$ and $\|\cdot\|_p$ denotes the usual norm in $L^p(\mathbb{R}^n)$. We use the following version of the Fourier transform. For $f \in L^1(\mathbb{R}^n)$ and $\omega \in \mathbb{R}^n$, let $\widehat{f}(\omega) = \int_{\mathbb{R}^n} f(x)e^{2\pi i\omega\cdot x}dx$. By Plancherel's Theorem, there is a unitary map $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ such that $\mathcal{F}f = \widehat{f}$, for all $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. We call both $\mathcal{F}$ and the map $f \mapsto \widehat{f}$ the Fourier transform. We will often use $\widehat{f}$ instead of $\mathcal{F}f$, for any $f \in L^2(\mathbb{R}^n)$.

If we consider $\mathbb{R}^n$ as a topological group with addition of vectors as the group operation, a subgroup $\mathcal{L}$ of $\mathbb{R}^n$ is called a lattice in $\mathbb{R}^n$ if $\mathcal{L}$ is discrete and $\mathbb{R}^n/\mathcal{L}$ is compact. The dual lattice of $\mathcal{L}$ is $\mathcal{L}^* = \{\nu \in \mathbb{R}^n : \nu \cdot k \in \mathbb{Z}\}$. That is, $\nu \in \mathbb{R}^n$ is in $\mathcal{L}^*$ if and only if $e^{2\pi i\nu\cdot k} = 1$, for all $k \in \mathcal{L}$. If $\mathcal{L}$ is a lattice in $\mathbb{R}^n$, then there exists an invertible $n \times n$ matrix B such that $\mathcal{L} = B\mathbb{Z}^n = \{Bk : k \in \mathbb{Z}^n\}$. Then $\mathcal{L}^* = (B^{-1})^t\mathbb{Z}^n$, where $(B^{-1})^t$ is the transpose of the inverse of B. Let $Q = \{(B^{-1})^t\theta : \theta \in \left[-\frac{1}{2}, \frac{1}{2}\right)^n\}$. Then $\mathbb{R}^n$ is the disjoint union of the $Q + \nu$, $\nu \in \mathcal{L}^*$. We equip Q with the restriction of Lebesgue measure.

We will need a form of the Poisson summation formula. For $g \in L^1(\mathbb{R}^n)$, the periodization $\mathcal{P}_{\mathcal{L}^*}g$ of g with respect to $\mathcal{L}^*$ is given by

$$\mathcal{P}_{\mathcal{L}^*}g(\theta) = \sum_{\nu\in\mathcal{L}^*} g(\theta+\nu),$$

for any $\theta \in \mathbb{R}^n$ for which the series converges.

Lemma 2.1. *Let $\mathcal{L}$ be a lattice in $\mathbb{R}^n$ and let $\mathcal{L}^*$ be its dual lattice. Let $g \in L^1(\mathbb{R}^n)$. Then $\sum_{\nu\in\mathcal{L}^*} g(\theta+\nu)$, converges for a.e. $\theta \in \mathbb{R}^n$ and $\mathcal{P}_{\mathcal{L}^*}g\big|_Q \in L^1(Q)$.*

Proof. We adapt the standard argument as in the proof of Theorem 8.31 of [10].

$$\begin{aligned}
\int_Q \sum_{\nu\in\mathcal{L}^*} |g(\theta+\nu)|d\theta &= \sum_{\nu\in\mathcal{L}^*}\int_Q |g(\theta+\nu)|d\theta \\
&= \sum_{\nu\in\mathcal{L}^*}\int_{Q+\nu} |g(\theta)|d\theta = \int_{\mathbb{R}^n} |g(\theta)|d\theta = \|g\|_1.
\end{aligned}$$

By the sum variation of the Dominated Convergence Theorem, $\sum_{\nu\in\mathcal{L}^*} g(\theta+\nu)$ converges for a.e. $\theta \in Q$ and $\mathcal{P}_{\mathcal{L}^*}g\big|_Q \in L^1(Q)$. Since $\mathcal{P}_{\mathcal{L}^*}g$ is $\mathcal{L}^*$-periodic, $\sum_{\nu\in\mathcal{L}^*} g(\theta+\nu)$ converges for a.e. $\theta \in \mathbb{R}^n$. □

Proposition 2.2. *Let $\mathcal{L}$ be a lattice in $\mathbb{R}^n$ and let $\mathcal{L}^*$ be its dual lattice. Let $g \in L^1(\mathbb{R}^n)$. For each $\ell \in \mathcal{L}$,*

$$\int_Q \mathcal{P}_{\mathcal{L}^*}g(\theta)e^{2\pi i\theta\cdot\ell}d\theta = \widehat{g}(\ell).$$

Moreover, if $\widehat{g}(\ell) = 0$, for all $\ell \in \mathcal{L}$, then $\mathcal{P}_{\mathcal{L}^*}g = 0$.

Proof. Again, dominated convergence justifies the interchange of sum and integral below. For $\ell \in \mathcal{L}$,

$$\begin{aligned}
\int_Q \mathcal{P}_{\mathcal{L}^*}g(\theta)e^{2\pi i\theta\cdot\ell}d\theta &= \int_Q \sum_{\nu\in\mathcal{L}^*} g(\theta+\nu)e^{2\pi i\theta\cdot\ell}d\theta = \sum_{\nu\in\mathcal{L}^*}\int_Q g(\theta+\nu)e^{2\pi i\theta\cdot\ell}d\theta \\
&= \sum_{\nu\in\mathcal{L}^*}\int_{Q+\nu} g(\theta)e^{2\pi i(\theta-\nu)\cdot\ell}d\theta = \sum_{\nu\in\mathcal{L}^*}\int_{Q+\nu} g(\theta)e^{2\pi i\theta\cdot\ell}d\theta \\
&= \int_{\mathbb{R}^n} g(\theta)e^{2\pi i\theta\cdot\ell}d\theta = \widehat{g}(\ell).
\end{aligned}$$

For the last statement in the proposition, we do a linear change of variables. Let $C = (B^{-1})^t$, where $\mathcal{L} = B\mathbb{Z}^n$. That is, C is the invertible matrix such that $\mathcal{L}^* = C\mathbb{Z}^n$. Now define $h \in L^1(\mathbb{R}^n)$ by $h(\omega) = g(C\omega)$, for a.e. $\omega \in \mathbb{R}^n$. Form the $\mathbb{Z}^n$ periodization of h. That is,

$$\mathcal{P}_{\mathbb{Z}^n} h(\omega) = \sum_{k\in\mathbb{Z}^n} h(\omega + k) = \sum_{k\in\mathbb{Z}^n} g(C\omega + Ck) = \mathcal{P}_{\mathcal{L}^*} g(C\omega),$$

which converges for a.e. $\omega \in \mathbb{R}^n$, and $\mathcal{P}_{\mathbb{Z}^n} h$ is integrable over $\left[-\frac{1}{2}, \frac{1}{2}\right)^n$. Let $(c_j)_{j\in\mathbb{Z}^n}$ be the Fourier multi-series of the periodic function $\mathcal{P}_{\mathbb{Z}^n} h$. For each $j \in \mathbb{Z}^n$, compute the Fourier coefficient c_j using the change of variables $\theta = C\omega$ and noting that $C^{-1} = B^t$. So

$$\begin{aligned} c_j &= \int_{\left[-\frac{1}{2},\frac{1}{2}\right)^n} \mathcal{P}_{\mathbb{Z}^n} h(\omega) e^{2\pi i\omega\cdot j} d\omega = \int_{\left[-\frac{1}{2},\frac{1}{2}\right)^n} \mathcal{P}_{\mathcal{L}^*} g(C\omega) e^{2\pi i\omega\cdot j} d\omega \\ &= |\det(B)| \int_Q \mathcal{P}_{\mathcal{L}^*} g(\theta) e^{2\pi i\theta\cdot Bj} d\theta = |\det(B)| \widehat{g}(Bj). \end{aligned}$$

Now $Bj \in \mathcal{L}^*$, for all $j \in \mathbb{Z}^n$. So, if $\widehat{g}(\ell) = 0$, for all $\ell \in \mathcal{L}^*$, then every Fourier coefficient of $\mathcal{P}_{\mathbb{Z}^n} h$ is 0. This implies $\mathcal{P}_{\mathbb{Z}^n} h = 0$, which implies $\mathcal{P}_{\mathcal{L}^*} g = 0$. □

Corollary 2.3. *With the notation of Proposition 2.2, let Y be an open subset of $\mathbb{R}^n$ such that $(Y+\nu)\cap(Y+\kappa) = \emptyset$ if $\nu, \kappa \in \mathcal{L}^*$ with $\nu \neq \kappa$. Let $g \in L^1(\mathbb{R}^n)$ be such that $g(\theta) = 0$, for a.e. $\theta \in \mathbb{R}^n \setminus Y$. If $\widehat{g}(\ell) = 0$, for all $\ell \in \mathcal{L}$, then $g = 0$.*

Proof. Under the assumption on g, $\mathcal{P}_{\mathcal{L}^*} g$ agrees with g on Y, so Proposition 2.2 implies $g = 0$. □

Remark 2.4. For each $\ell \in \mathcal{L}$, let $e_\ell(\theta) = e^{2\pi i\theta\cdot\ell}$, for all $\theta \in \mathbb{R}^n$. In the language of [6], Definition 2.2, $\mathcal{D} = \{e_\ell|_Y : \ell \in \mathcal{L}\}$ is a determining set for $L^1(Y)$.

The concept of a range function goes back to [14] and was used in [5] for the characterization of closed subspaces of $L^2(\mathbb{R}^n)$ that are shift-invariant. The treatment of range functions given in [6] is most useful for our purposes. For a Hilbert space $\mathcal{H}$, let $Gr(\mathcal{H})$ denote the set of all closed subspaces of $\mathcal{H}$, the Grassmannian of $\mathcal{H}$. If $\mathcal{K} \in Gr(\mathcal{H})$, let $P_\mathcal{K}$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$.

Definition 2.5. Let $\mathcal{H}$ be a separable Hilbert space and let $(\mathcal{X}, \Sigma)$ be a measurable space. A measurable *range function* for $\mathcal{H}$ based on $\mathcal{X}$, is a mapping $J : \mathcal{X} \to Gr(\mathcal{H})$ such that, for any $\xi, \eta \in \mathcal{H}$, the map $x \to \langle P_{J(x)}\xi, \eta\rangle$ is measurable.

In all the cases we consider here, $\mathcal{H}$ is a separable Hilbert space. A map $F : \mathcal{X} \to \mathcal{H}$ is measurable if $x \to \langle F(x), \eta\rangle$ is a measurable complex-valued function, for each $\eta \in \mathcal{H}$. If μ is a positive measure on $(\mathcal{X}, \Sigma)$, let

$$L^2(\mathcal{X}, \mu, \mathcal{H}) = \left\{ F : \mathcal{X} \to \mathcal{H} \middle| F \text{ is measurable and } \int_{\mathcal{X}} \|F(x)\|^2 d\mu(x) < \infty \right\},$$

with the usual identification of functions that agree μ-almost everywhere. If the measure μ is understood from context, we write $L^2(\mathcal{X}, \mathcal{H})$. The inner product of $F_1, F_2 \in L^2(\mathcal{X}, \mathcal{H})$ is given by

$$\langle F_1, F_2\rangle = \int_{\mathcal{X}} \langle F_1(x), F_2(x)\rangle d\mu(x).$$

Let J be a measurable range function for $\mathcal{H}$ based on $\mathcal{X}$. Then, for any $F \in L^2(\mathcal{X}, \mathcal{H})$, $x \to \langle \mathcal{P}_{J(x)} F(x), \eta\rangle$ is measurable, for every $\eta \in \mathcal{H}$. Given J, define $\mathcal{M}_J = \{F \in L^2(\mathcal{X}, \mathcal{H}) : F(x) \in J(x)$, for μ-a.e. $x \in \mathcal{X}\}$. The next Proposition gathers together Propositions 2.1, 2.2, and 2.3 of [6].

Proposition 2.6. *Let $J$and K be measurable range functions for $\mathcal{H}$ based on $\mathcal{X}$. Then*

(i) *$\mathcal{M}_J$ is a closed subspace of $L^2(\mathcal{X}, \mathcal{H})$,*

(ii) *if $\mathcal{P}_{\mathcal{M}_J}$ is the orthogonal projection of $L^2(\mathcal{X}, \mathcal{H})$ onto $\mathcal{M}_J$, then, for any $F \in L^2(\mathcal{X}, \mathcal{H})$, $(\mathcal{P}_{\mathcal{M}_J} F)(x) = \mathcal{P}_{J(x)} F(x)$, for a.e. $x \in \mathcal{X}$, and*

(iii) *$\mathcal{M}_J = \mathcal{M}_K$ if and only if $J(x) = K(x)$, for a.e. $x \in \mathcal{X}$.*

If $\mathcal{H}$ is a Hilbert space, let $\mathcal{U}(\mathcal{H})$ denote the group of all unitary operators on $\mathcal{H}$. If Λ is any discrete group, a unitary representation of Λ on $\mathcal{H}$ is a homomorphism $\sigma : \Lambda \to \mathcal{U}(\mathcal{H})$. Let σ_1 and σ_2 be unitary representations of Λ on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. If there exists a unitary map $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that $W\sigma_1(a) = \sigma_2(a)W$, for all $a \in \Lambda$, then σ_1 and σ_2 are called equivalent.

3 Crystal groups

Let O_n denote the compact group of orthogonal $n \times n$ real matrices. For $x \in \mathbb{R}^n$ and $A \in \mathrm{O}_n$, define $[x, A]$ to be the affine map, $z \to A(z + x)$, of $\mathbb{R}^n$. Let

$$\mathrm{Iso}_n(\mathbb{R}) = \{[x, A] : x \in \mathbb{R}^n, A \in \mathrm{O}_n\},$$

the group of all isometries of $\mathbb{R}^n$. The composition of isometries is the group product: For $[x, A], [y, B] \in \mathrm{Iso}_n(\mathbb{R})$, $[x, A][y, B] = [B^{-1}x + y, AB]$ and $[x, A]^{-1} = [-Ax, A^{-1}]$. Note that $[0, \mathrm{id}]$ is the identity element of $\mathrm{Iso}_n(\mathbb{R})$, where id denotes the identity $n \times n$ matrix. Given the product topology of $\mathbb{R}^n \times \mathrm{O}_n$, $\mathrm{Iso}_n(\mathbb{R})$ is a locally compact group. Let $\mathrm{Trans}_n = \{[x, \mathrm{id}] : x \in \mathbb{R}^n\}$, the set of pure translations in $\mathrm{Iso}_n(\mathbb{R})$. This is a closed normal subgroup of $\mathrm{Iso}_n(\mathbb{R})$. Let $q : \mathrm{Iso}_n(\mathbb{R}) \to \mathrm{O}_n$ be the homomorphism given by $q[x, A] = A$, for all $[x, A] \in \mathrm{Iso}_n(\mathbb{R})$.

A crystal group is a discrete subgroup Γ of $\mathrm{Iso}_n(\mathbb{R})$ such that $\mathbb{R}^n/\Gamma$ is compact, where $\mathbb{R}^n/\Gamma$ is the set of all Γ-orbits, with the quotient topology. Section 7.5 of [17] presents the basic properties of crystal groups. The translation subgroup of Γ is $T = \Gamma \cap \mathrm{Trans}_n$, which is a normal subgroup, and the point group is $\Pi = q(\Gamma)$, which is isomorphic to the quotient group Γ/T. Then Π is a finite subgroup of O_n, while T is a free abelian group of rank n. Indeed, $\mathcal{L} = \{\ell \in \mathbb{R}^n : [\ell, \mathrm{id}] \in T\}$ is a lattice in $\mathbb{R}^n$. For $[\ell, \mathrm{id}] \in T$ and $[x, M] \in \Gamma$,

$$[x, M][\ell, \mathrm{id}][x, M]^{-1} = [x + \ell, M][-Mx, M^{-1}] = [M\ell, \mathrm{id}].$$

This shows that if $\ell \in \mathcal{L}$, then $M\ell \in \mathcal{L}$, for all $M \in \Pi$.

Fix a cross-section $\gamma : \Pi \to \Gamma$ of the T-cosets in Γ; so $q\big(\gamma(M)\big) = M$, for all $M \in \Pi$. With γ fixed, for each $M \in \Pi$, let $x_M \in \mathbb{R}^n$ be such that $\gamma(M) = [x_M, M]$. Then $\Gamma = \{[\ell + x_M, M] : \ell \in \mathcal{L}, M \in \Pi\}$. For many crystal groups, $[0, M] \in \Gamma$, for all $M \in \Pi$. Then, choose $x_M = 0$, for each $M \in \Pi$. When this can be done, Γ is isomorphic to the semidirect product $\mathcal{L} \rtimes \Pi$. Such crystal groups are called symmorphic.

Lemma 3.1. *With the $x_M \in \mathbb{R}^n$ selected as above, $N^{-1}x_K + x_N - x_{KN} \in \mathcal{L}$, for all $K, N \in \Pi$.*

Proof. For $K, N \in \Pi$, calculate the following element of Γ:

$$\begin{aligned} [x_K, K][x_N, N][x_{KN}, KN]^{-1} &= [N^{-1}x_K + x_N, KN][-KNx_{KN}, (KN)^{-1}] \\ &= [Kx_K + KNx_N - KNx_{KN}, \mathrm{id}] \in T. \end{aligned}$$

Thus, $Kx_K + KNx_N - KNx_{KN} \in \mathcal{L}$. Then

$$N^{-1}x_K + x_N - x_{KN} = N^{-1}K^{-1}\big(Kx_K + KNx_N - KNx_{KN}\big)$$

is in $\mathcal{L}$ as well. $\square$

Associated with the lattice $\mathcal{L}$ is the dual lattice $\mathcal{L}^* = \{\nu \in \mathbb{R}^n : \nu \cdot \ell \in \mathbb{Z}, \text{ for all } \ell \in \mathcal{L}\}$. Since $M^t = M^{-1}$, for any $M \in \Pi \subseteq \mathrm{O}_n$, this implies $\mathcal{L}^*$ is left invariant when multiplied by members of Π. With this action of Π on $\mathcal{L}^*$, form the semidirect product $\mathcal{L}^* \rtimes \Pi = \{(\nu, M) : \nu \in \mathcal{L}^*, M \in \Pi\}$, with the group product given by

$$(\kappa, L)(\nu, M) = (M^{-1}\kappa + \nu, LM), \text{ for } (\kappa, L), (\nu, M) \in \mathcal{L}^* \rtimes \Pi.$$

This auxilliary group might be isomorphic to Γ, but this is not always the case. As with Γ, $\mathcal{L}^* \rtimes \Pi$ acts on $\mathbb{R}^n$. For $(\nu, M) \in \mathcal{L}^* \rtimes \Pi$ and $\omega \in \mathbb{R}^n$, let $(\nu, M) \cdot \omega = M(\omega + \nu)$. This identifies $\mathcal{L}^* \rtimes \Pi$ with a discrete group of isometries of $\mathbb{R}^n$ such that, since $\mathcal{L}^*$ is a full-rank lattice, $\mathbb{R}^n/(\mathcal{L}^* \rtimes \Pi)$ is compact. That is, $\mathcal{L}^* \rtimes \Pi$ is also a crystal group. Let $\Gamma^* = \mathcal{L}^* \rtimes \Pi$.

For any $\omega \in \mathbb{R}^n$, let $\Gamma^*\omega = \{M(\omega + \nu) : (\nu, M) \in \Gamma^*\}$, the Γ^*-orbit of ω, and let $\Gamma^*_\omega = \{(\nu, M) \in \Gamma^* : M(\omega + \nu) = \omega\}$, the stabilizer of ω. There exist points ω in $\mathbb{R}^n$ such that $\Gamma^*_\omega = \{(0, \mathrm{id})\}$ (see Theorem 6.6.12 of [17]). Fix $\omega_0 \in \mathbb{R}^n$ such that $\Gamma^*_{\omega_0} = \{(0, \mathrm{id})\}$. For each $(\nu, M) \in \Gamma^* \setminus \{(0, \mathrm{id})\}$, let $H_{(\nu,M)} = \{\omega \in \mathbb{R}^n : \|\omega - \omega_0\| < \|\omega - M(\omega_0 + \nu)\|\}$.

Definition 3.2. The *Dirichlet domain* for Γ^* containing ω_0 is

$$\Omega_{\omega_0} = \bigcap \{H_{(\nu,M)} : (\nu, M) \in \Gamma^*, (\nu, M) \neq (0, \mathrm{id})\}.$$

Let $A \subseteq \mathbb{R}^n$, then $A + \nu = \{\omega + \nu : \omega \in A\}$, for $\nu \in \mathcal{L}^*$, and $MA = \{M\omega : \omega \in A\}$, for $M \in \Pi$. Also, ∂A denotes the boundary of A.

Proposition 3.3. *Let $\omega_0 \in \mathbb{R}^n$ be such that $\Gamma^*_{\omega_0} = \{(0, \mathrm{id})\}$ and let $\Omega = \Omega_{\omega_0}$. Then Ω has the following properties:*

1. *Ω is open*
2. *Ω is convex*
3. *For $(\kappa, L), (\nu, M) \in \Gamma^*$, $(\kappa, L) \neq (\nu, M)$, $(L(\Omega + \kappa)) \cap (M(\Omega + \nu)) = \emptyset$*
4. *$\cup_{(\nu,M)\in\Gamma^*} M(\overline{\Omega} + \nu) = \mathbb{R}^n$*
5. *$\cup_{(\nu,M)\in\Gamma^*} M(\Omega + \nu)$ is a co-null subset of $\mathbb{R}^n$.*

Proof. Properties 1, 2, 3, and 4 are well-known (see, for example, Theorem 6.6.13 and the definition of *fundamental domain* on page 233 of [17]). Since Ω is open and convex $\partial\Omega$ has Lebesgue measure 0. Each map $\omega \to M(\omega+\nu)$ is an isometry, so $\partial(M(\Omega+\nu)) = M(\partial\Omega+\nu)$ is a null set, for each $(\nu, M) \in \Gamma^*$. Now $\mathbb{R}^n \setminus \cup_{(\nu,M)\in\Gamma^*} M(\Omega+\nu) \subseteq \cup_{(\nu,M)\in\Gamma^*} M(\partial\Omega+\nu)$, which is a null set since Γ^* is countable. This implies 5. $\square$

Let $\Pi\Omega = \cup_{M\in\Pi} M\Omega$. Then $\Pi\Omega$ is a fundamental domain for $\mathcal{L}^*$. In particular, we have the following corollary.

Corollary 3.4. *If $\nu_1, \nu_2 \in \mathcal{L}^*$, $\nu_1 \neq \nu_2$, then $(\Pi\Omega+\nu_1) \cap (\Pi\Omega+\nu_2) = \emptyset$ and $\cup_{\nu\in\mathcal{L}^*}(\Pi\Omega+\nu)$ is co-null in $\mathbb{R}^n$.*

4 Several Unitary Transformations

In order to illuminate aspects of the natural unitary representation of the crystal group Γ in the next section, the Hilbert space $L^2(\mathbb{R}^n)$ will be transformed into a number of different realizations by specific unitary maps. The first is the Fourier transform viewed as a unitary map of $L^2(\mathbb{R}^n)$ onto itself.

For the other Hilbert spaces that arise, we equip the open subsets Ω and $\Pi\Omega$ with the restriction of Lebesgue measure and $\Omega\times\Pi\times\mathcal{L}^*$ with the product measure, where Π and $\mathcal{L}^*$ have counting measure. We can use $\Omega\times\Pi\times\mathcal{L}^*$ to almost parametrize $\mathbb{R}^n$. Let $X = \cup_{(L,\nu)\in\Pi\times\mathcal{L}^*} L(\Omega+\nu)$. By Proposition 3.3, X is an open co-null subset of $\mathbb{R}^n$.

Lemma 4.1. *The map $\phi : \Omega\times\Pi\times\mathcal{L}^* \to X$ given by $\phi(\omega, L, \nu) = L(\omega+\nu)$ is a measure preserving homeomorphism.*

Proof. Routine. $\square$

For any $\theta \in X$, let $\omega_\theta \in \Omega$, $L_\theta \in \Pi$, and $\nu_\theta \in \mathcal{L}^*$ be such that $L_\theta(\omega_\theta+\nu_\theta) = \theta$. For each $\xi \in L^2(\mathbb{R}^n)$, let $W_1\xi = \xi\circ\phi$. Then $W_1\xi$ is a measurable function on $\Omega\times\Pi\times\mathcal{L}^*$.

Lemma 4.2. *The map W_1 is a unitary from $L^2(\mathbb{R}^n)$ onto $L^2(\Omega\times\Pi\times\mathcal{L}^*)$ and W_1^{-1} is given by, for $g \in L^2(\Omega\times\Pi\times\mathcal{L}^*)$, $W_1^{-1}g(\theta) = g(\omega_\theta, L_\theta, \nu_\theta)$, for a.e. $\theta \in X$.*

Proof. Also routine. $\square$

Our next unitary transformation is a simple unitary operator on the Hilbert space $L^2(\Omega \times \Pi \times \mathcal{L}^*)$. Recall that, for each $L \in \Pi$, $x_L \in \mathbb{R}^n$ was selected so that $[x_L, L] \in \Gamma$. Define a function $w_2 : \Omega \times \Pi \times \mathcal{L}^* \to \mathbb{T}$ by $w_2(\omega, L, \nu) = e^{-2\pi i\nu\cdot x_L}$, for all $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$. Pointwise multiplication by such a $\mathbb{T}$-valued continuous function is a unitary operator and we formulate this in a lemma.

Lemma 4.3. *For each $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$, let $W_2 g = w_2 g$. That is,*

$$(W_2 g)(\omega, L, \nu) = e^{-2\pi i\nu\cdot x_L} g(\omega, L, \nu),$$

for a.e. $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$. Then W_2 is a unitary operator on $L^2(\Omega \times \Pi \times \mathcal{L}^*)$ and $W_2^{-1} g = \overline{w_2} g$, for all $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$.

Remark 4.4. The function w_2 plays an important role in the characterization of Γ–shift invariant closed subspaces of $L^2(\mathbb{R}^n)$ obtained in Section 6, so it is important to note that this function is independent of the choice of the cross-section $\gamma : \Pi \to \Gamma$. Indeed, if γ' is a different cross-section, then, for each $L \in \Pi$, $\gamma'(L) = [x'_L, L]$, for some $x'_L \in \mathbb{R}^n$. But $[x_L, L]^{-1}[x'_L, L] = [x'_L - x_L, \mathrm{id}] \in T$, so $k = x'_L - x_L \in \mathcal{L}$. Thus, $e^{-2\pi i\nu\cdot x'_L} = e^{-2\pi i\nu\cdot(x_L+k)} = e^{-2\pi i\nu\cdot x_L}$.

Let $\{\delta_\nu : \nu \in \mathcal{L}^*\}$ be the orthonormal basis of $\ell^2(\mathcal{L}^*)$ consisting of the usual delta functions. We will use $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$, which is a separable Hilbert space. As we work with each of $L^2(\mathbb{R}^n)$, $L^2(\Omega \times \Pi \times \mathcal{L}^*)$, $\ell^2(\mathcal{L}^*)$, and $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$, any inner product that arises will be denoted $\langle\cdot,\cdot\rangle$ and the corresponding norm denoted by $\|\cdot\|$, relying on the reader to know which Hilbert space this inner product or norm belongs to from the context.

We now turn to the definition of a unitary transformation of $L^2(\Omega\times\Pi\times\mathcal{L}^*)$ into $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$. As above, for any $\theta \in \Pi\Omega$, there are unique $\omega_\theta \in \Omega$ and $L_\theta \in \Pi$ so that $L_\theta \omega_\theta = \theta$. For any $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$ and $\nu \in \mathcal{L}^*$, let $g_\nu(\theta) = g(\omega_\theta, L_\theta, \nu)$, for a.e. $\theta \in \Pi\Omega$. Then $g_\nu \in L^2(\Pi\Omega)$, for each $\nu \in \mathcal{L}^*$, and $\|g\|^2 = \sum_{\nu\in\mathcal{L}^*} \|g_\nu\|^2$. Define $W_3 g : \Pi\Omega \to \ell^2(\mathcal{L}^*)$ by

$$W_3 g(\theta) = \sum_{\nu\in\mathcal{L}^*} g_\nu(\theta)\delta_\nu, \text{ for a.e. } \theta \in \Pi\Omega. \tag{1}$$

Lemma 4.5. *If $W_3 g$ is defined by (1), then $W_3 g \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$, for each $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$ and W_3 is a unitary map of $L^2(\Omega \times \Pi \times \mathcal{L}^*)$ onto $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$. Moreover, for $F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$,*

$$W_3^{-1} F(\omega, L, \nu) = \langle F(L\omega), \delta_\nu\rangle, \tag{2}$$

for a.e. $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$.

Proof. For each $\kappa \in \mathcal{L}^*$, the map $\theta \mapsto \langle W_3 g(\theta), \delta_\kappa \rangle = g_\kappa(\theta)$ is measurable. Since $\{\delta_\kappa : \kappa \in \mathcal{L}^*\}$ is a countable basis of $\ell^2(\mathcal{L}^*)$, $W_3 g$ is measurable. Moreover, using (1)

$$\int_{\Pi\Omega} \|W_3 g(\theta)\|^2 d\theta = \int_{\Pi\Omega} \sum_{\nu \in \mathcal{L}^*} |g_\nu(\theta)|^2 d\theta = \sum_{\nu \in \mathcal{L}^*} \int_{\Pi\Omega} |g_\nu(\theta)|^2 d\theta = \|g\|^2. \quad (3)$$

Therefore, $W_3 g \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$ and W_3 is clearly linear. By (3), W_3 is an isometry. If $F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$ and (2) is used to define $W_3^{-1}F$ on $\Omega \times \Pi \times \mathcal{L}^*$, then one verifies, with an argument similar to (3) in reverse, that $W_3^{-1}F$ is square-integrable on $\Omega \times \Pi \times \mathcal{L}^*$ and $W_3 W_3^{-1} F = F$. □

5 The natural representation

The action of Γ on $\mathbb{R}^n$ generates a unitary representation. For $[x, M] \in \Gamma$, $\pi[x, M]$ is the unitary operator given by, for $f \in L^2(\mathbb{R}^n)$,

$$\pi[x, M]f(z) = f([x, M]^{-1}z) = f(M^{-1}z - x), \text{ for a.e. } z \in \mathbb{R}^n.$$

It is routine to show that π is a homomorphism of Γ into $\mathcal{U}\big(L^2(\mathbb{R}^n)\big)$. As mentioned in the introduction, we will call π the *natural representation* of Γ. The Fourier transform, as a unitary map $\mathcal{F}$ on $L^2(\mathbb{R}^n)$, intertwines π with an equivalent representation $\widehat{\pi}$ by $\widehat{\pi}[\ell, M] = \mathcal{F}\pi[\ell, M]\mathcal{F}^{-1}$. A brief calculation, using the fact that M^{-1} agrees with the transpose matrix of $M \in \mathrm{O}_n$, shows that, for $\xi \in L^2(\mathbb{R}^n)$,

$$\widehat{\pi}[x, M]\xi(\omega) = e^{2\pi i (M^{-1}\omega)\cdot x} \xi(M^{-1}\omega), \text{ for a.e. } \omega \in \mathbb{R}^n. \quad (4)$$

Recall from Lemma 4.2 that $W_1\xi(\omega, L, \nu) = \xi\big(L(\omega + \nu)\big)$, for almost every $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$, for each $\xi \in L^2(\mathbb{R}^n)$, defines a unitary map of $L^2(\mathbb{R}^n)$ onto $L^2(\Omega \times \Pi \times \mathcal{L}^*)$. Let $\widehat{\pi}_1$ be the representation of Γ on $L^2(\Omega \times \Pi \times \mathcal{L}^*)$ given by $\widehat{\pi}_1[x, M] = W_1 \widehat{\pi}[x, M] W_1^{-1}$, for $[x, M] \in \Gamma$. For $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$, let $\xi = W_1^{-1} g$. Then, for $[x, M] \in \Gamma$ and a.e. $\omega \in \Omega$, $L \in \Pi$, and $\nu \in \mathcal{L}^*$.

$$\widehat{\pi}_1[x, M]g(\omega, L, \nu) = \widehat{\pi}[x, M]\xi\big(L(\omega + \nu)\big) = e^{2\pi i (M^{-1}L(\omega+\nu))\cdot x} \xi\big(M^{-1}L(\omega + \nu)\big).$$

Setting $\theta = M^{-1}L(\omega + \nu)$, note that $\omega_\theta = \omega$, $L_\theta = M^{-1}L$, and $\nu_\theta = \nu$. Thus,

$$\widehat{\pi}_1[x, M]g(\omega, L, \nu) = e^{2\pi i (M^{-1}L(\omega+\nu))\cdot x} g(\omega, M^{-1}L, \nu),$$

for a.e. $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$. Since $[x, M] \in \Gamma$, there exists $\ell \in \mathcal{L}$ so that $x = \ell + x_M$. For any $\nu \in \mathcal{L}^*$ and $N \in \Pi$, $e^{2\pi i (N\nu)\cdot \ell} = 1$. Therefore, $e^{2\pi i (M^{-1}L(\omega+\nu))\cdot x}$ can be written as $e^{2\pi i (M^{-1}L\nu)\cdot x_M} e^{2\pi i (M^{-1}L\omega)\cdot x} = e^{2\pi i \nu \cdot (L^{-1}Mx_M)} e^{2\pi i (M^{-1}L\omega)\cdot x}$ and

$$\widehat{\pi}_1[x, M]g(\omega, L, \nu) = e^{2\pi i \nu \cdot (L^{-1}Mx_M)} e^{2\pi i (M^{-1}L\omega)\cdot x} g(\omega, M^{-1}L, \nu). \tag{5}$$

We can use the rather elementary, but strategic, unitary W_2 from Lemma 4.3 to eliminate the first factor in the right hand side of (5). Recall that $W_2 g = w_2 g$, for all $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$, where $w_2(\omega, L, \nu) = e^{-2\pi i \nu \cdot x_L}$, for $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$. Let $\widehat{\pi}_2[x, M] = W_2 \widehat{\pi}_1[x, M] W_2^{-1}$, for all $[x, M] \in \Gamma$. Then, for $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$,

$$\begin{aligned}
&\widehat{\pi}_2[x, M]g(\omega, L, \nu) = e^{-2\pi i \nu \cdot x_L} \widehat{\pi}_1[x, M] W_2^{-1} g(\omega, L, \nu) \\
&= e^{-2\pi i \nu \cdot x_L} e^{2\pi i \nu \cdot (L^{-1}Mx_M)} e^{2\pi i (M^{-1}L\omega)\cdot x} W_2^{-1} g(\omega, M^{-1}L, \nu) \\
&= e^{-2\pi i \nu \cdot x_L} e^{2\pi i \nu \cdot (L^{-1}Mx_M)} e^{2\pi i (M^{-1}L\omega)\cdot x} e^{2\pi i \nu \cdot x_{M^{-1}L}} g(\omega, M^{-1}L, \nu).
\end{aligned}$$

The scalar in the previous line can be rearranged to

$$e^{2\pi i \nu \cdot (L^{-1}Mx_M + x_{M^{-1}L} - x_L)} e^{2\pi i (L\omega)\cdot (Mx)}.$$

Consider the expression $L^{-1}Mx_M + x_{M^{-1}L} - x_L$. By Lemma 3.1 with $K = M$ and $N = M^{-1}L$, we have that $L^{-1}Mx_M + x_{M^{-1}L} - x_L = \ell$, for some $\ell \in \mathcal{L}$. Since $\nu \in \mathcal{L}^*$, $e^{2\pi i \nu \cdot (L^{-1}Mx_M + x_{M^{-1}L} - x_L)} = 1$. Finally, for $[x, M] \in \Gamma$ and $g \in L^2(\Omega \times \Pi \times \mathcal{L}^*)$,

$$\widehat{\pi}_2[x, M]g(\omega, L, \nu) = e^{2\pi i (L\omega)\cdot (Mx)} g(\omega, M^{-1}L, \nu) \tag{6}$$

for a.e. $\omega \in \Omega$, $L \in \Pi$ and $\nu \in \mathcal{L}^*$.

The next step is to conjugate by the unitary $W_3 : L^2(\Omega \times \Pi \times \mathcal{L}^*) \to L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$ from Lemma 4.5. Let $\widehat{\pi}_3[x, M] = W_3 \widehat{\pi}_2[x, M] W_3^{-1}$, for all $[x, M] \in \Gamma$. For $F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$,

$$\begin{aligned}
\widehat{\pi}_3[x, M]F(\theta) &= W_3 \widehat{\pi}_2[x, M] W_3^{-1} F(\theta) \\
&= \sum_{\nu \in \mathcal{L}^*} \big(\widehat{\pi}_2[x, M] W_3^{-1} F(\omega_\theta, L_\theta, \nu)\big)\, \delta_\nu \\
&= e^{2\pi i (L_\theta \omega_\theta)\cdot (Mx)} \sum_{\nu \in \mathcal{L}^*} W_3^{-1} F(\omega_\theta, M^{-1}L_\theta, \nu) \delta_\nu. \qquad (7)
\end{aligned}$$

Using (2) we get $\sum_{\nu \in \mathcal{L}^*} W_3^{-1} F(\omega_\theta, M^{-1}L_\theta, \nu)\delta_\nu = F(M^{-1}L_\theta \omega_\theta)$. Recall that $L_\theta \omega_\theta = \theta$ and $M^t = M^{-1}$, so (7) implies

$$\widehat{\pi}_3[x, M]F(\theta) = e^{2\pi i (M^{-1}\theta)\cdot x} F(M^{-1}\theta), \text{ for a.e. } \theta \in \Pi\Omega, \tag{8}$$

and each $[x, M] \in \Gamma$ and any $F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$. We summarize these considerations in a proposition.

Proposition 5.1. *If $\widehat{\pi}_2[x, M]$ is defined by (6) and $\widehat{\pi}_3[x, M]$ is defined by (8), for each $[x, M] \in \Gamma$, then $\widehat{\pi}_2$ and $\widehat{\pi}_3$ are unitary representations of Γ that are each unitarily equivalent to the natural representation π.*

6 Invariant closed subspaces

Definition 6.1. Let $J : \Pi\Omega \to Gr\big(\ell^2(\mathcal{L}^*)\big)$ be a measurable range function. We say J is Π-invariant, if $J(L\theta) = J(\theta)$, for a.e. $\theta \in \Pi\Omega$ and every $L \in \Pi$.

Proposition 6.2. *Let $\mathcal{K}$ be a closed subspace of $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$. Then $\mathcal{K}$ is $\widehat{\pi}_3$-invariant if and only if there exists a Π-invariant measurable range function, J, for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$ such that $\mathcal{K} = \mathcal{M}_J$.*

Proof. First, suppose J is a Π-invariant measurable range function for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$ and $\mathcal{K} = \mathcal{M}_J$. Let $[x, M] \in \Gamma$. For any $F \in \mathcal{M}_J$, we know that $F(\theta) \in J(\theta)$, for a.e. $\theta \in \Pi\Omega$, and J is Π-invariant. This implies

$$\widehat{\pi}_3[x, M]F(\theta) = e^{2\pi i\theta\cdot(Mx)}F(M^{-1}\theta) \in J(M^{-1}\theta) = J(\theta),$$

for a.e. $\theta \in \Pi\Omega$. Thus, $\widehat{\pi}_3[x, M]F \in \mathcal{K}$, for all $F \in \mathcal{K}$ and $[x, M] \in \Gamma$. That is, $\mathcal{K}$ is $\widehat{\pi}_3$-invariant.

Now suppose $\mathcal{K}$ is a $\widehat{\pi}_3$-invariant closed subspace of $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$. By Remark 2.4, $\mathcal{D} = \{e_\ell|_{\Pi\Omega} : \ell \in \mathcal{L}\}$ is a determining set for $L^1(\Pi\Omega)$, where $e_\ell(\theta) = e^{2\pi i\theta\cdot\ell}$, for all $\theta \in \mathbb{R}^n$. If $F \in \mathcal{K}$ and $\ell \in \mathcal{L}$, then $[\ell, \mathrm{id}] \in T \subseteq \Gamma$ and (8) says

$$\widehat{\pi}_3[\ell, \mathrm{id}]F(\theta) = e^{2\pi i\theta\cdot\ell}F(\theta) = e_\ell(\theta)F(\theta), \text{ for a.e. } \theta \in \Pi\Omega.$$

Therefore, $e_\ell F \in \mathcal{K}$, for each $\ell \in \mathcal{L}$. In the language of [6], $\mathcal{K}$ is multiplicatively-invariant with respect to $\mathcal{D}$, denoted $\mathcal{D} - MI$. By Theorem 2.4 of [6], there exists a measurable range function $J : \Pi\Omega \to Gr\big(\ell^2(\mathcal{L}^*)\big)$ so that

$$\mathcal{K} = \mathcal{M}_J = \{F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big) : F(\theta) \in J(\theta), \text{ for a.e. } \theta \in \Pi\Omega\}.$$

Moreover, J is unique up to almost everywhere agreement and, if $\mathcal{A}$ is any countable dense subset of $\mathcal{K}$, $J(\theta)$ is the closed linear span of $\{\varphi(\theta) : \varphi \in \mathcal{A}\}$, for a.e. $\theta \in \Pi\Omega$. Fix such an $\mathcal{A}$.

Fix $L \in \Pi$ so that $[x_L, L] \in \Gamma$ and use (8) again to get

$$\widehat{\pi}_3[x_L, L]F(L\omega) = e^{2\pi i\omega\cdot(x_L)}F(\omega), \text{ for a.e. } \omega \in \Omega. \tag{9}$$

For $\varphi \in \mathcal{A}$, let $\varphi^L = \widehat{\pi}_3[x_L, L]\varphi$ and let $L \cdot \mathcal{A} = \{\varphi^L : \varphi \in \mathcal{A}\}$. Then $L \cdot \mathcal{A}$ is a dense subset of $\mathcal{K}$. Therefore, by (9), for a.e. $\omega \in \Omega$,

$$J(L\omega) = \overline{\operatorname{span}}\{\varphi^L(L\omega) : \varphi \in \mathcal{A}\} = \overline{\operatorname{span}}\{\varphi(\omega) : \varphi \in \mathcal{A}\} = J(\omega).$$

Since $L \in \Pi$ is arbitrary, J is Π-invariant. □

Before formulating the main result, we recall the various ingredients: Γ is a crystal group in dimension n with point group Π and associated lattice $\mathcal{L}$. The dual lattice of $\mathcal{L}$ is $\mathcal{L}^*$. There is an open subset Ω of $\mathbb{R}^n$ described in Proposition 3.3 and the collection of sets $\{L(\Omega + \nu) : L \in \Pi, \nu \in \mathcal{L}^*\}$ tiles $\mathbb{R}^n$ up to a null set.

Theorem 6.3. *Let $\mathcal{V}$ be a closed subspace of $L^2(\mathbb{R}^n)$. Then $\mathcal{V}$ is invariant under shifts by elements of Γ if and only if there exists a Π-invariant measurable range function J for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$ such that, for any $f \in \mathcal{V}$, there exists $F \in \mathcal{M}_J$ so that, for almost every $\omega \in \Omega$, every $L \in \Pi$, and every $\nu \in \mathcal{L}^*$,*

$$\widehat{f}\big(L(\omega + \nu)\big) = e^{2\pi i\nu\cdot x_L}\langle F(L\omega), \delta_\nu\rangle. \tag{10}$$

Proof. Suppose that $\mathcal{V}$ is a closed subspace of $L^2(\mathbb{R}^n)$ that is invariant under shifts by elements of Γ. That is, $\mathcal{V}$ is π-invariant. Let $U = W_3 \circ W_2 \circ W_1 \circ \mathcal{F}$, a unitary map that intertwines the natural representation π with $\widehat{\pi}_3$. Therefore, $U\mathcal{V}$ is a $\widehat{\pi}_3$-invariant closed subspace of $L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$. By Proposition 6.2, there exists a Π-invariant measurable range function $J : \Pi\Omega \to Gr\big(\ell^2(\mathcal{L}^*)\big)$ such that $U\mathcal{V} = \mathcal{M}_J$. For any $f \in \mathcal{V}$, let $F = Uf$. This can be written as $W_1\widehat{f} = W_2^{-1}W_3^{-1}F$, which is an element of $L^2(\Omega \times \Pi \times \mathcal{L}^*)$. By the definition of W_1 given just before Lemma 4.2,

$$W_1\widehat{f}(\omega, L, \nu) = \widehat{f}\big(L(\omega + \nu)\big), \tag{11}$$

for a.e. $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$. On the other hand, using the equations for W_2^{-1} and W_3^{-1} given in Lemmas 4.3 and 4.5, we have

$$W_2^{-1}W_3^{-1}F(\omega, L, \nu) = e^{2\pi i\nu\cdot x_L}\langle F(L\omega), \delta_\nu\rangle, \tag{12}$$

for a.e. $(\omega, L, \nu) \in \Omega \times \Pi \times \mathcal{L}^*$. Comparing (11) with (12) yields (10).

Conversely, suppose J is a Π-invariant measurable range function for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$ and that $\mathcal{V}$ is related to $\mathcal{M}_J$ via (10). Then $\mathcal{M}_J$ is $\widehat{\pi}_3$-invariant by Proposition 6.2 and unpacking (10) shows that $U^{-1}\mathcal{M}_J = \mathcal{V}$. Since U is the intertwining unitary between π and $\widehat{\pi}_3$, $\mathcal{V}$ must be π-invariant. Thus, $\mathcal{V}$ is invariant under shifts by elements of Γ. □

Theorem 6.3 is a direct generalization of Proposition 1.5 in [5], where it is attributed to Helson [14]. It is not surprising that the range function associated to a π-invariant closed subspace must be Π-invariant. However, the appearance of the factor $e^{2\pi i\nu\cdot x_L}$ in (10) is not so obvious. As noted earlier, for a given $L \in \Pi$, the choice of x_L such that $[x_L, L] \in \Gamma$ is not unique. However, if $y \in \mathbb{R}^n$ also satisfies $[y, L] \in \Gamma$, then $x_L - y \in \mathcal{L}$. Thus, $e^{2\pi i\nu\cdot y} = e^{2\pi i\nu\cdot x_L}$, for all $\nu \in \mathcal{L}^*$. To help understand the role this term plays, we present an example of a group, necessarily non-symmorphic, where this factor is nontrivial.

Example 6.4. The patch of brick wall illustrated in Figure 1 is meant to be a region of a pattern extending in all directions. The symmetry group of this pattern is often denoted pg, so set $\Gamma = pg$. If the origin is placed at the bottom of one of the vertical line segments, let $\mathbf{u}$ denote a horizontal vector pointing right whose length equals the length of one brick. Let $\mathbf{v}$ be an upward pointing vector whose length equals twice the width of a brick. Then, we can take the translation lattice of Γ to be $\mathcal{L} = \{j\mathbf{u}+k\mathbf{v} : j, k \in \mathbb{Z}\}$. Let $\sigma \in \mathrm{O}_2$ denote the reflection about the horizontal axis. This is not a symmetry of the pattern, but $[\frac{1}{2}\mathbf{u}, \sigma] \in \Gamma$. Indeed, the point group of Γ is $\Pi = \{\mathrm{id}, \sigma\}$, isomorphic to $\mathbb{Z}_2$. If $T = \{[j\mathbf{u} + k\mathbf{v}, \mathrm{id}] : j, k \in \mathbb{Z}\}$ is the translation subgroup of Γ, we can take the cross-section of the T-cosets to be given by $\gamma(\mathrm{id}) = [0, \mathrm{id}]$ and $\gamma(\sigma) = [\frac{1}{2}\mathbf{u}, \sigma]$.

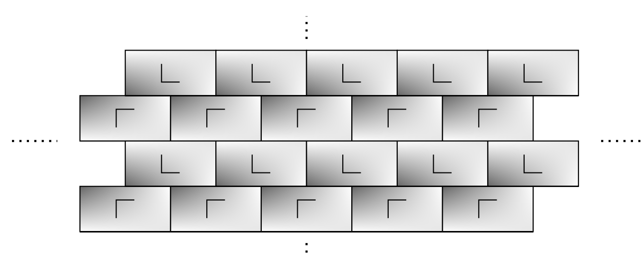

Figure 1: A pattern illustrating the symmetries of the wallpaper group pg.

Fix the basis for $\mathbb{R}^2$ to be $\{\mathbf{u}, \mathbf{v}\}$ and represent elements $x \in \mathbb{R}^2$ as $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then $\mathcal{L} = \left\{ \ell = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} : \ell_1, \ell_2 \in \mathbb{Z} \right\} = \mathbb{Z}^2$, viewed as a lattice in $\mathbb{R}^2$. So $\mathcal{L}^* = \mathbb{Z}^2$, as well. We set $\Gamma^* = \mathcal{L}^* \rtimes \Pi$. Let $\omega_0 = \begin{pmatrix} 0 \\ 1/4 \end{pmatrix}$ and form Ω as in Definition 3.2. Then $\Omega = \left\{ \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} ; -\frac{1}{2} < \omega_1 < \frac{1}{2}, 0 < \omega_2 < \frac{1}{2} \right\}$ as illustrated in Figure 2. The open set $\sigma\Omega$ is outlined as well. Note that $\Pi\Omega = \Omega \cup \sigma\Omega$.

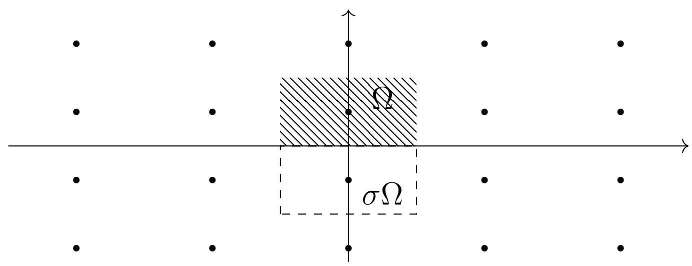

Figure 2: The Γ^*-orbit of ω_0 and domain Ω.

Let's construct a simple Π-invariant measurable range function for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$. Let $\kappa = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathcal{L}^*$ and consider the one-dimensional subspace $V = \mathbb{C}(\delta_0 + \delta_\kappa) = \{\alpha(\delta_0 + \delta_\kappa) : \alpha \in \mathbb{C}\}$ of $\ell^2(\mathcal{L}^*)$. Let E be a measurable subset of Ω as pictured in Figure 3 where we have surpressed the vertical axis and changed the scale a little.

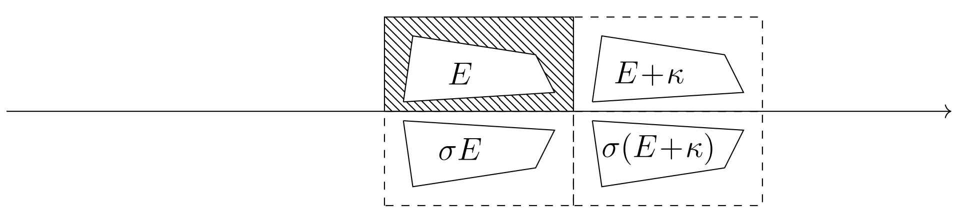

Figure 3: The measurable subset E of Ω and three of its shifts.

Now consider the map $J : \Pi\Omega \to Gr\big(\ell^2(\mathcal{L}^*)\big)$ given by $J(\theta) = V$, for $\theta \in E \cup \sigma E$, and $J(\theta) = \{0\}$, for $\theta \in \Pi\Omega \setminus (E \cup \sigma E)$. Then, $\mathcal{M}_J$ consists of all $F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$ such that $F(\theta) = 0$, for a.e. $\theta \in \Pi\Omega \setminus (E \cup \sigma E)$, and $F(\theta) \in \mathbb{C}(\delta_0 + \delta_\kappa)$, for a.e. $\theta \in E \cup \sigma E$. Before looking at (10) in this situation, note that $x_{\mathrm{id}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $x_\sigma = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$. So, $e^{2\pi i \nu \cdot x_{\mathrm{id}}} = 1$ and $e^{2\pi i \nu \cdot x_\sigma} = (-1)^{\nu_1}$, for all $\nu = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \in \mathcal{L}^*$. Consider the following three properties for an $f \in L^2(\mathbb{R}^2)$:

(a) $\widehat{f}(\omega + \kappa) = \widehat{f}(\omega)$, for a.e. $\omega \in \Omega$.

(b) $\widehat{f}\big(\sigma(\omega + \kappa)\big) = -\widehat{f}\big(\sigma(\omega)\big)$, for a.e. $\omega \in \Omega$.

(c) $\widehat{f}(\theta) = 0$, for all $\theta \in \mathbb{R}^2 \setminus \big(E \cup (E + \kappa) \cup \sigma E \cup \sigma(E + \kappa)\big)$.

What Theorem 6.3 implies for the simple range function J is that, if $\mathcal{V}$ denotes the set of all $f \in L^2(\mathbb{R}^2)$ whose Fourier transform $\widehat{f}$ satisfies (a), (b), and (c), then $\mathcal{V}$ is a closed subspace of $L^2(\mathbb{R}^2)$ that is invariant under shifts from the wallpaper group pg.

Although the range function used in Example 6.4 is simple compared to the complexity that is possible, the example does suggest that the statement of Theorem 6.3 can be refined. We return to Γ denoting a crystal group in dimension n with Π, $\mathcal{L}$, and Ω as before. For each $L \in \Pi$, define a unitary operator U_L on $\ell^2(\mathcal{L}^*)$ by $U_L h(\nu) = e^{2\pi i \nu \cdot x_L} h(\nu)$, for all $\nu \in \mathcal{L}^*$ and for each $h \in \ell^2(\mathcal{L}^*)$. In Theorem 6.3, we used range functions based on $\Pi\Omega$ that are Π-invariant, so completely determined by their restriction to Ω. We now give a version where Π-invariance is replaced with twisting by the unitaries U_L, $L \in \Pi$.

Definition 6.5. For a measurable range function $J : \Omega \to Gr\big(\ell^2(\mathcal{L}^*)\big)$, let $J^\Gamma : \Pi\Omega \to Gr\big(\ell^2(\mathcal{L}^*)\big)$ be given by $J^\Gamma(L\omega) = U_L J(\omega)$, for each $\omega \in \Omega$ and all $L \in \Pi$.

If J is a measurable range function for $\ell^2(\mathcal{L}^*)$ based on Ω, then it is clear that J^Γ is a measurable range function for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$. Note that we can create a Π-invariant range function J' from J based on $\Pi\Omega$ by simply letting $J'(L\omega) = J(\omega)$, for all $L \in \Pi$, $\omega \in \Omega$. We call J' the Π-invariant extension of J. The following lemma is immediate from the definitions.

Lemma 6.6. *Let J be a measurable range function for $\ell^2(\mathcal{L}^*)$ based on Ω and let J' be the Π-invariant extension. For any $G \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$, we have $G(L\omega) \in J'(L\omega)$ if and only if $U_L\big(G(L\omega)\big) \in J^\Gamma(L\omega)$, for all $\omega \in \Omega$ and $L \in \Pi$.*

We can now present a modified statement of Theorem 6.3 as a corollary.

Corollary 6.7. *Let $\mathcal{V}$ be a closed subspace of $L^2(\mathbb{R}^n)$. Then $\mathcal{V}$ is invariant under shifts by elements of Γ if and only if there exists a measurable range function J for $\ell^2(\mathcal{L}^*)$ based on Ω such that, for any $f \in \mathcal{V}$, there exists $F \in \mathcal{M}_{J^\Gamma}$ so that $\widehat{f}\big(L(\omega+\nu)\big) = \langle F(L\omega), \delta_\nu\rangle$, for almost every $\omega \in \Omega$, every $L \in \Pi$, and every $\nu \in \mathcal{L}^*$.*

Proof. Suppose that $\mathcal{V}$ is invariant under shifts by Γ, and let K be a Π-invariant range function for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$, as described in Theorem 6.3. Thus for any $f \in \mathcal{V}$, there exists $G \in \mathcal{M}_K$ such that $\widehat{f}\big(L(\omega+\nu)\big) = e^{2\pi i\nu\cdot x_L}\langle G(L\omega), \delta_\nu\rangle$ for a.e. $\omega \in \Omega$ and all $L \in \Pi$ and $\nu \in \mathcal{L}^*$. Let J denote the restriction of K to Ω. Then $J' = K$. Define $F \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$ by $F(L\omega) = U_L\big(G(L\omega)\big)$, for a.e. $\omega \in \Omega$ and any $L \in \Pi$. By Lemma 6.6, $F \in \mathcal{M}_{J^\Gamma}$. And

$$\widehat{f}\big(L(\omega+\nu)\big) = e^{2\pi i\nu\cdot x_L}\langle G(L\omega), \delta_\nu\rangle = \langle U_L\big(G(L\omega)\big), \delta_\nu\rangle = \langle F(L\omega), \delta_\nu\rangle, \quad (13)$$

for a.e. $\omega \in \Omega$, any $L \in \Pi$, and any $\nu \in \mathcal{L}^*$.

Conversely, suppose that for any $f \in \mathcal{V}$, there exists $F \in \mathcal{M}_{J^\Gamma}$ such that

$$\widehat{f}\big(L(\omega+\nu)\big) = \langle F(L\omega), \delta_\nu\rangle$$

for a.e. ω and all $L \in \Pi$, $\nu \in \mathcal{L}^*$. Defining $G \in L^2\big(\Pi\Omega, \ell^2(\mathcal{L}^*)\big)$ by $G(L\omega) = U_L^{-1}F(L\omega)$ for a.e. $\omega \in \Omega$, it is immediate that $\langle F(L\omega), \delta_\nu\rangle = e^{2\pi i\nu\cdot x_L}\langle G(L\omega), \delta_\nu\rangle$, so that

$$\widehat{f}\big(L(\omega+\nu)\big) = e^{2\pi i\nu\cdot x_L}\langle G(L\omega), \delta_\nu\rangle.$$

We show that $G \in \mathcal{M}_K$, where K is a Π-invariant measurable range function for $\ell^2(\mathcal{L}^*)$ based on $\Pi\Omega$. Since $F \in \mathcal{M}_{J^\Gamma}$ by assumption, we have

$$U_L G(L\omega) = F(L\omega) \in J^\Gamma(L\omega),$$

which by Lemma 6.6 is equivalent to $G(L\omega) \in J'(L\omega)$ for a.e. ω, where J' is the Π-invariant extension of J. Thus $G \in \mathcal{M}_{J'}$. It follows now from Theorem 6.3 that $\mathcal{V}$ is invariant under shifts by Γ. □

References

[1] H. Bacry, L. Michel and J. Zak: Symmetry and Classification of Energy Bands in Crystals. In: Doebner, HD., Hennig, JD., Palev, T.D. (eds)

Group Theoretical Methods in Physics. Lecture Notes in Physics, 313. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0012290, 1988.

[2] L.W. Baggett, A. Carey, Wm. Moran, and P. Ohring: General existence theorems for orthonormal wavelets, an abstract approach. *Publ. Res. Inst. Math. Sci.* 31:95-111, 1995.

[3] J.D. Blanchard: Minimally Supported Frequency Composite Dilation Wavelets. *Journal of Fourier Analysis and Applications* 15:796–815, 2009.

[4] J.D. Blanchard and K.R. Steffen: Crystallographic Haar-type Composite Dilation Wavelets. In: Cohen, J., Zayed, A. (eds) *Wavelets and Multiscale Analysis*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston. 83–108, 2011.

[5] M. Bownik: The Structure of Shift-Invariant Subspaces of $L^2(\mathbb{R}^n)$. *Journal of Functional Analysis* 177:282–309, 2000.

[6] M. Bownik and K.A. Ross: The Structure of Translation-Invariant Spaces on Locally Compact Abelian Groups. *Journal of Fourier Analysis and Applications* 21:849–884, 2015.

[7] J. Cano and B. Bradlyn: Band Representations and Topological Quantum Chemistry. *Annual Review of Condensed Matter Physics* 12:225–246, 2021.

[8] J. Cano, B. Bradlyn, Z. Wang, L. Elcoro, M.G. Vergniory, C. Felser, M.I. Aroyo, and B.A. Bernevig: Building blocks of topological quantum chemistry: Elementary band representations. *Physical Review* B 97: DOI: 10.1103/PhysRevB.97.035139, 2018

[9] I. Daubechies: *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, 1992.

[10] G.B. Folland: *Real Analysis, Modern Techniques and Their Applications, 2nd edition*, ohn Wiley & Sons, 1999.

[11] A.L. González and M.d.C. Moure: Crystallographic Haar Wavelets. *Journal of Fourier Analysis and Applications* 17: 1119–1137.

[12] K. Guo, D. Labate, W–Q Lim, G. Weiss and E. Wilson: Wavelets with Composite Dilations. *Electron. Res. Announc. Amer. Math. Soc.* 10:78–87, 2004.

[13] K. Guo, D. Labate, W–Q Lim, G. Weiss and E. Wilson: Wavelets with Composite Dilations and their MRA Properties. *Applied and Computational Harmonic Analysis* 20:202–236, 2006.

[14] H. Helson: *Lectures on Invariant Subspaces*, Academic Press, 1964.

[15] E. Kaniuth and K.F. Taylor: *Induced Representations of Locally Compact Groups*, Cambridge University Press, 2013.

[16] J. MacArthur and K.F. Taylor: Wavelets with Crystal Symmetry Shifts. *Journal of Fourier Analysis and Applications* 17:1109–1118, 2011.

[17] J.G. Ratcliffe: *Foundations of Hyperbolic Manifolds, Third Edition*, Springer-Nature, 2019.

[18] A. Ron and Z. Shen: Frames and stable bases for shift-invariant subspaces of $L_2(\mathbb{R}^d)$. *Canadian Journal of Mathematics* 47:1051–1094.